Logarithms Done Proper

A lograthim is a function

$$\mathrm{Log} : (0, \infty) \longrightarrow \mathbb{R}$$

satisfying

- $\mathrm{Log}(xy) = \mathrm{Log}(x) + \mathrm{Log}(y)$
- $\mathrm{Log}(x^n) = n\,\mathrm{Log}(x)$

There should be an associated function

$$\mathrm{Exp} : \mathbb{R} \longrightarrow (0, \infty)$$

satisfying

$$\mathrm{Log}(\mathrm{Exp}(x)) = x.$$

When I was in primary school, to divide

$$7.89123$$

by

$$3.142$$

I'd first calculate

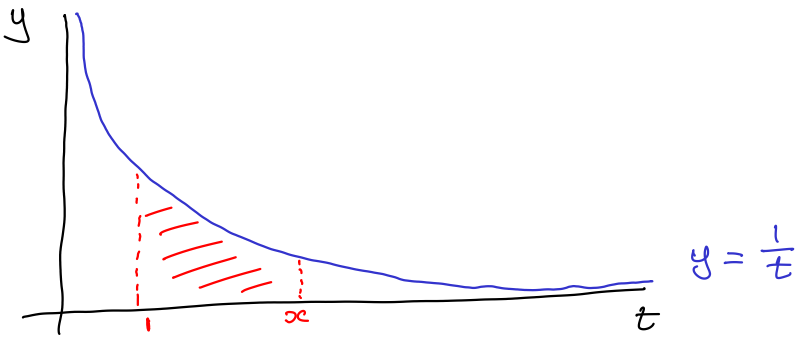
$$a = \mathrm{Log}_{10}(7.89123)$$

$$b = \mathrm{Log}_{10}(3.142)$$

and then I'd look up

$$\mathrm{Exp}(a-b) = \frac{7.89123}{3.142}$$

Defn For $x > 0$ let A be the area



between the curve $y = \frac{1}{t}$ and the t-axis from $t=1$ to $t=x$.

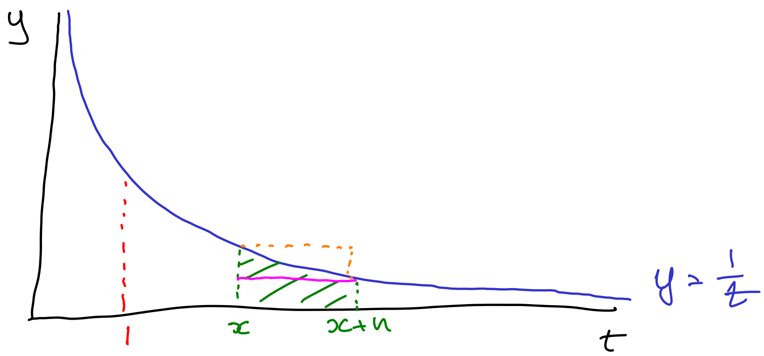
We define

$$\ln(x) = \begin{cases} A & \text{if } x \geq 1 \\ -A & \text{if } 0 < x < 1 \end{cases}$$

We call $\ln(x)$ the natural logarithm.

Theorem If $x > 0$ then $\frac{d}{dx}\ln(x) = \frac{1}{x}$.

Proof



$$\frac{h}{x+h} < \begin{matrix}\text{shaded} \\ \text{green} \\ \text{area}\end{matrix} < \frac{h}{x}$$

Divide throughout by h

$$\frac{1}{x+h} < \frac{\ln(x+h) - \ln(x)}{h} < \frac{1}{x}$$

So

$$\lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h} = \frac{1}{x}.$$

Sandwich Lemma

Thus

$$\frac{d}{dx} \ln(x) = \frac{1}{x}.$$

QED

Consequence 1 For $x, y > 0$ we have

$$\ln(xy) = \ln(x) + \ln(y).$$

Proof

$$\frac{d}{dx}\left(\ln(xy) - \ln(x)\right)$$

$$= \frac{d}{dx}\ln(xy) - \frac{d}{dx}\ln(x)$$

$$= \frac{\cancel{y}}{x\cancel{y}} - \frac{1}{x}$$

$$= 0$$

Thus

$$\ln(xy) - \ln(x) = c \qquad (*)$$

where c is a constant not depending on x.

Put, for instance, $x=1$ in $(*)$

$$\ln(y) - 0 = c.$$

Thus, from $(*)$, we get

$$\ln(xy) = \ln(x) + \ln(y).$$

QED

It is also easy to show that

$$\boxed{\ln(x^n) = n\ln(x)}$$

Past exam question

Find the derivative y' of

$$y = \frac{(x+1)(x+2)(x+3)}{x+4}.$$

Sol^n

$$\ln(y) = \ln(x+1) + \ln(x+2) + \ln(x+3) - \ln(x+4)$$

Differentiate both sides with respect to x:

$$\frac{1}{y}y' = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}$$

$$y' = \frac{(x+1)(x+2)(x+3)}{x+4}\left\{\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}\right\}.$$