Problem The population of teenagers in a village is infected with a virus. Each week 20% of the healthy teens test positive, and 30% of the infected teens become healthy.

There are 500 teens in the village, of which 100 are initially infected.

Determine the number of infected teenagers after 1, 2, 3, ... days, and investigate what happens in the long term.

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x_n = number of healthy teens at week n

y_n = " " infected " " " n

$x_0 = 400$

$y_0 = 100$

$$x_n = 0.8\, x_{n-1} + 0.3\, y_{n-1}$$

$$y_n = 0.2\, x_{n-1} + 0.7\, y_{n-1}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \underbrace{\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}}_{A} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \qquad (\dagger)$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} 400 \\ 100 \end{pmatrix} = \begin{pmatrix} 350 \\ 150 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} 350 \\ 150 \end{pmatrix} = \begin{pmatrix} 325 \\ 175 \end{pmatrix}$$

$$\vdots$$

From yesterday, we have

$$T^{-1} A T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

if T is a invertible 2×2 matrix whose columns are eigenvectors of A with corresponding eigenvalues λ_1, λ_2.

Then

$$A = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}$$

$$A^n = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cancel{T^{-1}} \cancel{T} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cancel{T^{-1}} \cdots \cancel{T} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}$$

$$\boxed{A^n = T \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} T^{-1}} \qquad (*)$$

Let's find eigenvalues of $A = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$.

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{pmatrix} = 0$$

$$(0.8-\lambda)(0.7-\lambda) - (0.2)(0.3) = 0$$

$$\lambda^2 - 1.5\lambda + 0.56 - 0.06 = 0$$

$$\lambda^2 - 1.5\lambda + 0.5 = 0.$$

$$(\lambda - 1)(\lambda - \tfrac{1}{2}) = 0$$

So eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$.

From $(\dagger)$ and $(*)$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} \underset{(**)}{=} A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = T \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} T^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

or

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \overbrace{T \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2^n} \end{pmatrix} T^{-1}}^{A^n} \begin{pmatrix} 400 \\ 100 \end{pmatrix}$$

For large n (using $(**)$)

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \underbrace{A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}_{\substack{\text{eigenvector with} \\ \text{eigenvalue } \lambda = 1.}} = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Let's find an eigenvector for $\lambda_1 = 1$.

$$(A - \lambda_1 I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

One such eigenvector is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 300 \\ 200 \end{pmatrix}$$

<u>Conclusion</u> In the long run there will be 300 healthy teens in the village and 200 infected teens every week.

$\boxed{\text{MARKOV PROCESS}}$