Rabbits

$F_0 = 1,\ F_1 = 1,\ F_2 = 2,\ F_3 = 3,\ F_4 = 5,\ F_5 = 8,\ F_6 = 13,\ \ldots$

where
$$\boxed{F_{n+1} = F_n + F_{n-1}}\ .$$

Aim: Find an explicit formula for F_n in terms of n but not involving F_{n-1} and F_{n-2}.

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}} \qquad (*)$$

But how do we calculate the n^{th} power of a matrix?

Theorem If a 2×2 matrix A has eigenvalues λ_1, λ_2 with corresponding eigenvectors v_1, v_2, and if the matrix

$$T = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$

is invertible, then

$$T^{-1} A T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Proof

$$\begin{aligned} T^{-1} A T &= T^{-1} A \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \\ &= T^{-1} \begin{pmatrix} | & | \\ Av_1 & Av_2 \\ | & | \end{pmatrix} \\ &= T^{-1} \begin{pmatrix} | & | \\ \lambda_1 v_1 & \lambda_2 v_2 \\ | & | \end{pmatrix} \\ &= T^{-1} \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\ &= T^{-1} T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \end{aligned}$$

Think!

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 12 \\ 15 & 24 \end{pmatrix}$$

$$= \begin{pmatrix} 5\cdot 1 & 6\cdot 2 \\ 5\cdot 3 & 6\cdot 4 \end{pmatrix}$$

QED

If $$T^{-1} A T = \underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}_{D}$$

Then

$$A = T D T^{-1}.$$

So

$$A^n = T D \cancel{T^{-1}} \cdot \cancel{T} D \cancel{T^{-1}} \cdot \cancel{T} D \cancel{T^{-1}} \cdot \ \cdots \ \cdot \cancel{T} D T^{-1}$$

$$= T D^n T^{-1}$$

$$\boxed{A^n = T \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} T^{-1}}$$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^3 = \begin{pmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{pmatrix}$$

Example Consider

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$P_A(\lambda) = \det(A - \lambda I)$$

$$= \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$$

$$= (1-\lambda)(-\lambda) - 1$$

$$= \lambda^2 - \lambda - 1.$$

The roots of $\lambda^2 - \lambda - 1$ are

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2}.$$

Golden Ratio

The roots are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} = \phi, \quad \lambda_2 = \frac{1-\sqrt{5}}{2} = \bar{\phi}$$

Note:

$$\phi \bar{\phi} = -1$$

Let's find corresponding eigenvectors.

Need to solve

$$(A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for $\lambda = \phi$ and $\lambda = \bar{\phi}$.

<u>Case $\lambda = \phi$</u>

$$\begin{pmatrix} 1-\phi & 1 \\ 1 & -\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

One solution

$$\begin{pmatrix} 1-\phi & 1 \\ 1 & -\phi \end{pmatrix}\begin{pmatrix} \phi \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

eigenvector for $\lambda = \phi$.

<u>Case $\lambda = \bar{\phi}$</u>

$$\begin{pmatrix} 1-\bar{\phi} & 1 \\ 1 & -\bar{\phi} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

One solution is

$$\begin{pmatrix} 1-\bar{\phi} & 1 \\ 1 & -\bar{\phi} \end{pmatrix}\begin{pmatrix} 1 \\ -\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

eigenvector for $\lambda = \bar{\phi}$.

$$T = \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = T \begin{pmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{pmatrix} T^{-1}$$

$$A = T \quad D \quad T^{-1}$$

$$A^n = T \quad D^n \quad T^{-1}$$

From $(*)$

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \overbrace{\begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix} \begin{pmatrix} \phi^n & 0 \\ 0 & \bar{\phi}^n \end{pmatrix} \begin{pmatrix} \phi & 1 \\ 1 & -\phi \end{pmatrix}^{-1}}^{A^n} \overbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}^{\begin{pmatrix} F_1 \\ F_0 \end{pmatrix}}$$

Exercise:

$$F_n = \frac{1}{\sqrt{5}} \phi^n - \frac{1}{\sqrt{5}} \bar{\phi}^n$$