Calculating Eigenvalues & Eigenvectors

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix of real numbers.

A non-zero vector $v = \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is said to be an eigenvector of A if

$$Av = \lambda v$$

for some number λ. We call λ the eigenvalue of A corresponding to v.

To calculate eigenvalues/vectors we need the following result.

Proposition Let A be a 2×2 matrix of real numbers, and let $v = \begin{pmatrix} x \\ y \end{pmatrix}$ be some non-zero vector, and suppose that

$$Av = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (*)$$

Then $\det(A) = 0$.

Proof If A^{-1} existed then, from $(*)$, we get

$$A^{-1}Av = A^{-1}v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so

$$v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We conclude that, under the hypothesis of the proposition, A^{-1} does not exist.

Recall

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Since A^{-1} does not exist, and since $\operatorname{adj}(A)$ always exists, we get

$$\det(A) = 0.$$

QED

How can we find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} ?$$

Suppose $v = \begin{pmatrix} x \\ y \end{pmatrix}$ is some eigenvector of A. Then

$$Av = \lambda v$$

for some λ and $v \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$Av = \lambda v$$

$$Av - \lambda v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$Av - \lambda I v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A - \lambda I) v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence $\boxed{\det(A - \lambda I) = 0}$ by the above proposition.

So $\det(A - \lambda I) = 0$

$$P_A(\lambda) = 0$$

$$\det\left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$$

$$\det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(1-\lambda) - 2.2 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda + 1)(\lambda - 3) = 0$$

Hence the eigenvalues of A are

$\lambda = -1$, $\lambda = 3$.

Now let's find corresponding eigenvectors.

Case $\lambda = -1$

Need $A v = \lambda v$

$A v = - v$

or $A v + v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $(A + I) v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $\left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

e.g. $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of A

with eigenvalue $\lambda = -1$.

Case $\lambda = 3$

$$A v = \lambda v$$

$$(A - \lambda I) v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

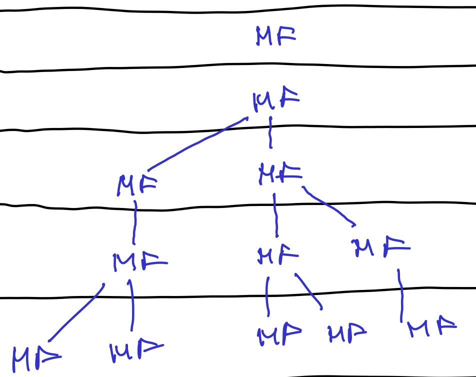
$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A

with eigenvalue $\lambda = 3$.

Rabbit Population

- One newly born male rabbit and one newly born female rabbit placed in a field.
- Rabbits can mate at age one month, and one month later the female produces one male/female pair of kittens.
- Rabbits never die.
- How many rabbits will there be in the field after 100 months.



month	# pairs
0	1
1	2
2	2
3	3
4	5
5	8
6	13
7	21
	⋮

Let F_n = number of pairs of rabbits after n months.

$$\boxed{F_{n+2} = F_{n+1} + F_n}$$

Fibonacci