Fourth online homework due Fri 27 Nov.
I'll extend the deadline a bit.

Eigenvalues & Eigenvectors

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix of real numbers.

$\boxed{\begin{array}{l} \textbf{Defn} \quad \text{A non-zero vector} \\ \qquad v = \begin{pmatrix} x \\ y \end{pmatrix} \\ \text{is an } \underline{\text{eigenvector}} \text{ of } A \text{ if there exists} \\ \text{some real number } \lambda \text{ such that} \\ \qquad Av = \lambda v. \\ \text{We call } \lambda \text{ the } \underline{\text{eigenvalue}} \text{ of } A \\ \text{corresponding to } v. \end{array}}$

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Consider

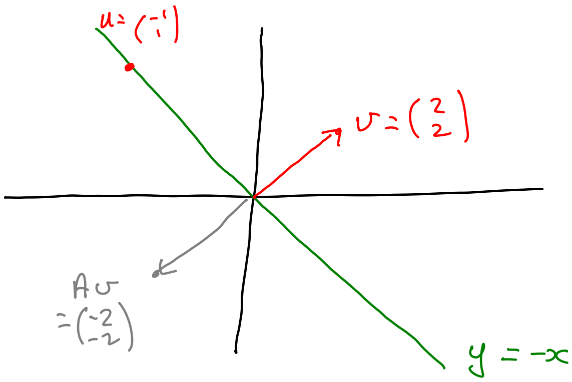
$$v = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

Then

$$\underset{A}{\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}}\underset{v}{\begin{pmatrix}4\\ 4\end{pmatrix}} = \begin{pmatrix}12\\ 12\end{pmatrix} = \underset{\lambda}{3}\underset{v}{\begin{pmatrix}4\\ 4\end{pmatrix}}$$

Then v is an eigenvector of A with eigenvalue $\lambda = 3$.

<u>Example</u> Let A be the 2×2 matrix of reflection in the line $y = -x$.



$$\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 2\end{pmatrix}$$

$Au = u$. So $u = \begin{pmatrix}-1\\ 1\end{pmatrix}$ is an eigenvector for A with eigenvalue $\lambda = 1$.

Also $Av = -v$ for $v = \begin{pmatrix}2\\ 2\end{pmatrix}$. Hence $v = \begin{pmatrix}2\\ 2\end{pmatrix}$ is an eigenvector for A with eigenvalue $\lambda = -1$.

<u>Example</u> Give me a matrix A that has <u>no</u> eigenvectors.

Answer: Let A be the matrix of rotation about the origin through an angle θ with $\theta \neq 0, \pi$. Then "clearly" A has no eigenvector.

Let A be a 2×2 matrix.

Defn The polynomial

$$P_A(\lambda) = \det(A - \lambda I)$$

is called the characteristic polynomial of A

$3x^7 - 2x^2 + 5$

$5x^{10} - 3x^7 + x + 1$

$5\lambda^{10} - 3\lambda^7 + \lambda + 1$

Example $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$P_A(\lambda) = \det\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$$

$$= (2-\lambda)(2-\lambda) - 1\cdot 1$$

$$P_A(\lambda) = \lambda^2 - 4\lambda + 3$$

Note:

$$P_A(\lambda) = \lambda^2 - 4\lambda + 3$$

$$P_A(2) = 4 - 8 + 3 = -1$$

$$P_A(A) = A^2 - 4A + 3I$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - 4\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Cayley - Hamilton Theorem

For any $n \times n$ matrix A we have

$$P_A(A) = 0I$$