Determinants

(Mainly 2x2 matrices)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Defn The adjoint matrix of A is

$$\mathrm{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Observation

$$A \cdot \mathrm{adj}(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Defn The determinant of the 2x2 matrix A is the number

$$\det(A) = ad - bc.$$

Note

$$A \cdot \mathrm{adj}(A) = \det(A) \cdot I$$

or $$A\left(\frac{1}{\det(A)}\,\mathrm{adj}(A)\right) = I.$$

Thus:

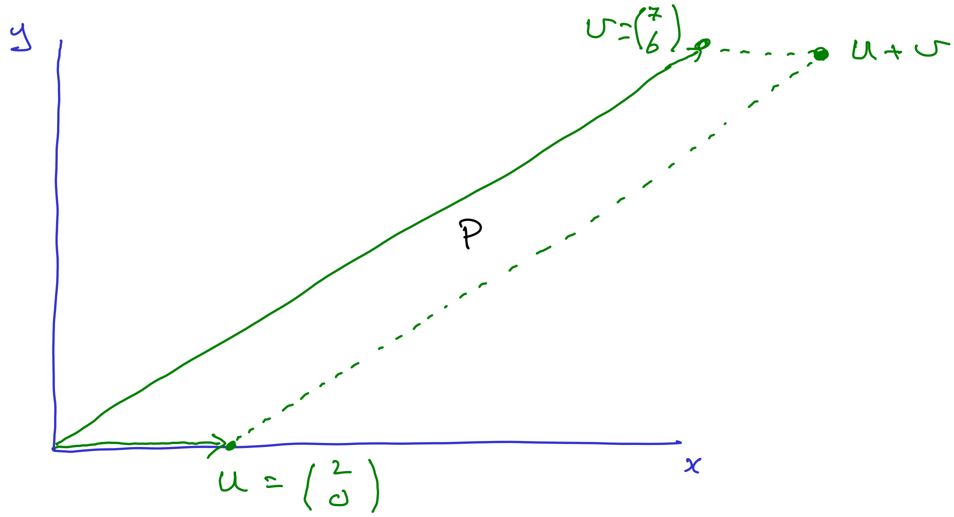
Proposition If $\det(A) \neq 0$ then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

for any 2×2 matrix.

Consider two "random" vectors

$$u = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad v = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$



So u and v determine a parallelogram P.

Area of P = base $\times$ $\perp^r$ height

$= 2 \times 6 = 12$.

Consider

$$A = \begin{pmatrix} 2 & 7 \\ 0 & 6 \end{pmatrix}$$

The vectors u, v can be thought of as the columns of a matrix A.

$$\det(A) = 2.6 - 0.7 = 12$$

Theorem 1 The determinant of a 2×2 matrix is equal to ± the area of the parallelogram determined by its two columns

Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

$$\det(A) = 1.4 - 3.2 = -2$$

$$\det(B) = 5.8 - 7.6 = -2$$

$$\det(AB) = \det \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

$$= 19.50 - 43.22 = 4.$$

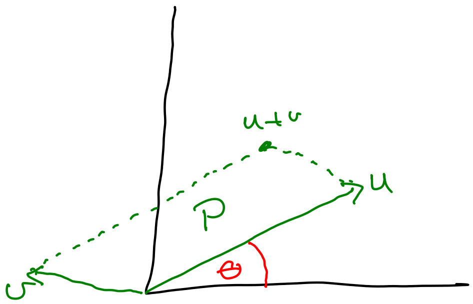
Note
$$\det(AB) = \det(A) \times \det(B)$$
in this example.

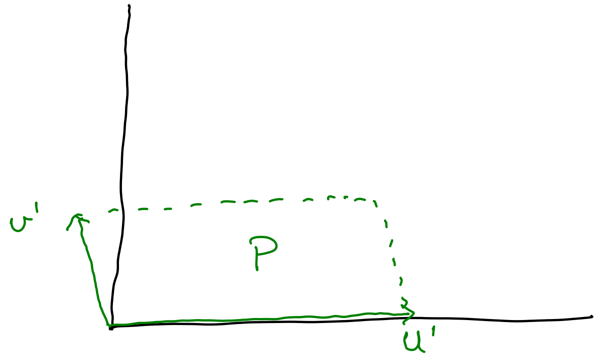
This example generalizes to

Theorem 2: For any 2×2 matrices A, B we have

$$\det(AB) = \det(A)\det(B)$$

Towards a proof that areas of parallelograms are captured by determinants.





$$u = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} u'$$

$$v = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} v'$$

$$\det \begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}$$

$$= \det \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} | & | \\ u' & v' \\ | & | \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \det \begin{pmatrix} | & | \\ u' & v' \\ | & | \end{pmatrix}$$

$$= (\cos^2\theta + \sin^2\theta)\,(\pm \text{Area of } P)$$

$$= \pm \text{Area of } P.$$

QED.

So we've proved Theorem 1.