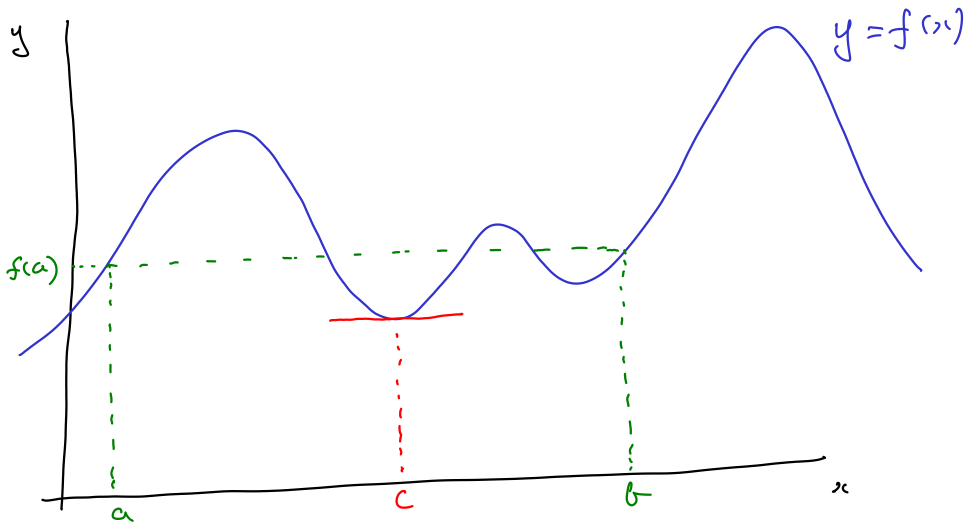
Rolle's Theorem

Suppose $f(x)$

- is continuous on $[a, b]$
- is differentiable on (a, b)
- and $f(a) = f(b)$.

Then there exists at least one number $c \in (a, b)$ such that $f'(c) = 0$.



Exercise Prove that there is exactly one solution to the equation

$$x^3 + x + 1 = 0. \qquad (*)$$

Proof

Let $f(x) = x^3 + x + 1$.

Note that f is differentiable at all points of $\mathbb{R}$.

$f(-1) < 0$

$f(1) > 0$

So the Intermediate Value Theorem implies that there is at least one $c \in [-1, 1]$ such that $f(c) = 0$. Thus there is at least one solution to $(*)$.

Note $f'(x) = 3x^2 + 1$.

Note also that $f'(x) > 0$ for all $x \in \mathbb{R}$.

If there were two solutions to $(*)$, say

$f(a) = f(b) = 0$,

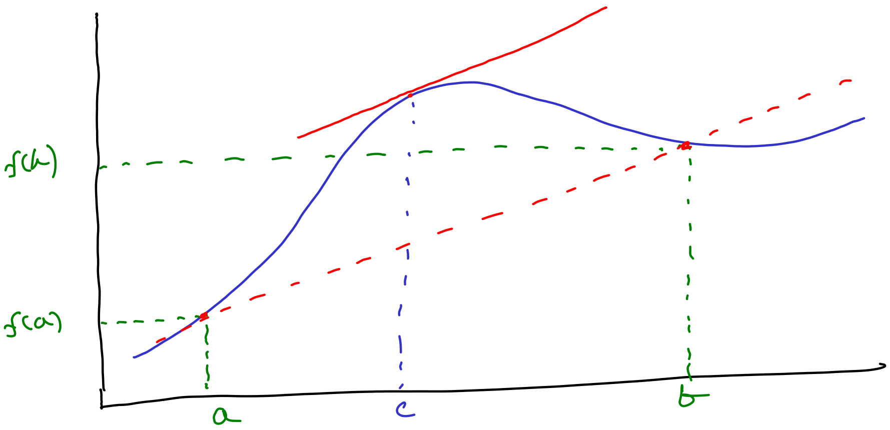
then by Rolle's Theorem we'd have $f'(x) = 0$ for some x. But $f'(x) > 0$ for all x.

Hence $(*)$ has at most one solution.

Thus $(*)$ has exactly one solution.

Q.E.D.

The Mean Value Theorem



<u>Theorem</u> (MVT) Suppose that

$f : [a,b] \longrightarrow \mathbb{R}$ is

- continuous on $[a,b]$
- differentiable on (a,b).

Then there exists at least one number $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} .$$

Logarithms	& Exponents
$4^2 = 16$	$\log_2 16 = 4$
$4^3 = 64$	$\log_4 64 = 3$
$4^{\frac{1}{2}} = \sqrt{4} = 2$	$\log_4 2 = \frac{1}{2}$
$4^{-2} = \frac{1}{4^2} = \frac{1}{16}$	$\log_4 \frac{1}{16} = -2$
$4^{\frac{3}{2}} = (4^{\frac{1}{2}})^3 = 2^3 = 8$	$\log_4 8 = \frac{3}{2}$
$4^{\frac{5}{2}} = (4^{\frac{1}{2}})^5 = 2^5 = 32$	$\log_4 32 = \frac{5}{2}$

We can make sense of 4^n when n is an integer $n = 0, \pm 1, \pm 2, \pm 3, \ldots,$ and when n is a fraction:

$$4^{\frac{p}{q}} = \left(\sqrt[q]{4}\right)^p .$$

Terminology: fractions of the form p/q with p, q integers are called rational numbers.

Q: What does $3^{\sqrt{2}}$ mean?

The issue is that $\sqrt{2} \neq p/q$ for any integers p, q.

We'd like to treat exponents, and logarithms carefully so that we have a meaning $3^{\sqrt{2}}$ (and not just think of "chuck it into a calculator").

Is $(\sqrt{2})^{\sqrt{2}}$ rational or not?

I have no idea!