Review of important terminology

- $f(x)$ is <u>continuous</u> at $x=c$ iff $f(c)$ is defined and $\lim_{x \to c} f(x) = f(c)$.

- $f(x)$ is <u>differentiable</u> at $x=c$ iff $f(c)$ is defined and the limit
$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$
exists.

- $x=c$ is a <u>critical point</u> of a function $f(x)$ iff $f(c)$ is defined and either $f'(c)$ does not exist or $f'(c)=0$.

- $f(x)$ is <u>increasing</u> on (a,b) if $f'(x) \geq 0$ for all $x \in (a,b)$.

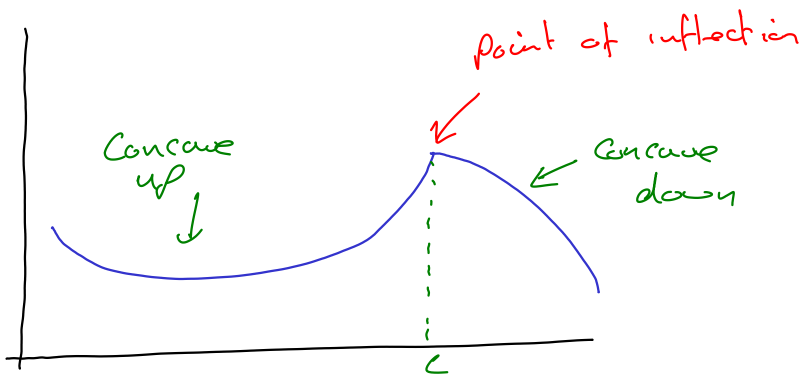
- $f(x)$ is <u>concave up</u> on (a,b) if $f''(x) \geq 0$ for all $x \in (a,b)$.

- $f(x)$ is <u>concave down</u> on (a,b) if $f''(x) \leq 0$ for all $x \in (a,b)$.

- $x=c$ is a <u>point of inflection</u> of $f(x)$ if concavity changes at $x=c$.

if and only if

iff



Theorem If $f(x)$ is differentiable at $x=c$ then $f(x)$ is continuous at $x=c$.

Proof Suppose $f(x)$ is differentiable at $x=c$.

This means

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = l \quad \text{exists}.$$

This implies

$$\left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}\right) = \left(\lim_{h\to 0} h\right) l$$

Thus

$$\lim_{h\to 0}\left(\frac{\cancel{h}\,(f(c+h)-f(c))}{\cancel{h}}\right) = 0$$

Therefore

$$\lim_{h\to 0}\left(f(c+h)-f(c)\right) = 0$$

We then have

$$\lim_{h\to 0} f(c+h) - \lim_{h\to 0} f(c) = 0$$

Then

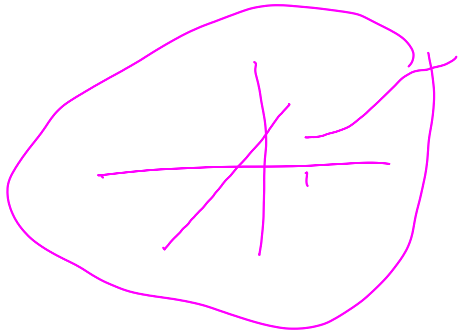
$$\lim_{h\to 0} f(c+h) = f(c)$$

or

$$\lim_{x\to c} f(x) = f(c).$$

QED.

Example Find all possible values of a and b such that



$$f(x) = \begin{cases} x^2+x+1 & , \quad x \geq 1 \\ ax+b & , \quad x<1 \end{cases}$$

is differentiable at all points.

Sol^n $f(x)$ is "clearly" differentiable at all points $x \neq 1$.

Need to choose a and b such that $f(x)$ is differentiable at $x=1$.

In particular $f(x)$ must be continuous at $x=1$. i.e.

$$\lim_{x \to 1} f(x) = f(1).$$

This means

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = f(1)$$

$$\lim_{x \to 1-} ax+b = 3$$

and $\boxed{a+b = 3}$. Let's assume $a+b=3$.

Now $f'(1)$ exists means

$$\lim_{h \to 0} \frac{f(1+h)-f(1)}{h} \quad \text{exists}$$

$$\iff \lim_{h \to 0-} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0+} \frac{f(1+h)-f(1)}{h}$$

$\iff \quad q'(1) = p'(1)$

where $q(x) = ax+b$ $\qquad p(x) = x^2+x+1$

$q'(1) = a \quad = \quad p'(1) = 3$

So we need $\underline{\underline{a = 3}}$.

$\underline{\underline{b = 0}}$.