$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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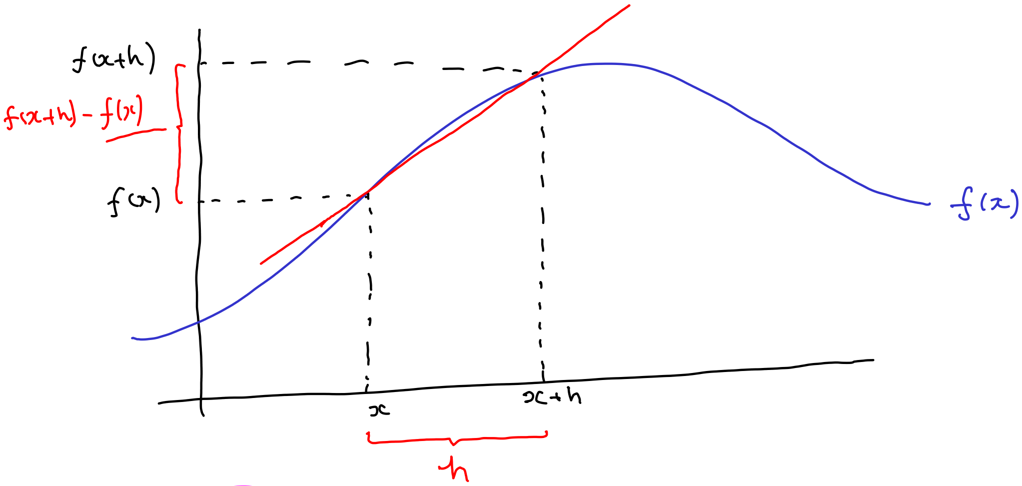

$$\frac{f(x+h) - f(x)}{h} = \text{slope of the red line}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ = slope of the tangent line to the curve $y = f(x)$ at the point x

Remember: $\frac{d}{dx} f(x) = f'(x)$

Rules of differentiation

$$\boxed{\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)}$$ Sum Rule

Proof

LHS =

$$\lim_{h\to 0} \frac{\left(f(x+h) + g(x+h)\right) - \left(f(x) + g(x)\right)}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right) + \lim_{h\to 0}\left(\frac{g(x+h) - g(x)}{h}\right)$$

$$= \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$ QED

Example $\frac{d}{dx}\left(x^{\frac{3}{2}} + \sin x\right)$

$$= \frac{d}{dx}x^{\frac{3}{2}} + \frac{d}{dx}\sin x$$

$$= \frac{3}{2}x^{\frac{1}{2}} + \cos(x).$$

$$\boxed{\frac{d}{dx}\, k f(x) = k \frac{d}{dx} f(x)}$$ (here $k \in \mathbb{R}$ is any constant)

Scalar product Rule

<u>Example</u>

$$\frac{d}{dx}\left(3e^{x}\right) = 3\frac{d}{dx}e^{x} = 3e^{x}.$$

$$\frac{d}{dx}\left(f(x)\,g(x)\right) = \left(\frac{d}{dx}f(x)\right)g(x) + f(x)\left(\frac{d}{dx}g(x)\right)$$

Product Rule

$d(uv) = u\,dv + v\,du$

Example

$$\frac{d}{dx}\left(x^2 \sin x\right) = \left(\frac{d}{dx}x^2\right)\sin(x) + x^2\left(\frac{d}{dx}\sin x\right)$$

$$= 2x \sin x + x^2 \cos x \,.$$

Example $\quad y = (x^2+1)(x^3+2)$

$$\frac{dy}{dx} = \left(\frac{d}{dx}(x^2+1)\right)(x^3+2) + (x^2+1)\left(\frac{d}{dx}(x^3+2)\right)$$

$$= 2x(x^3+2) + (x^2+1)\,3x^2$$

$$= 5x^4 + 3x^2 + 4x \,.$$

Chain Rule

Given two functions $f(x)$ and $g(x)$ we can consider the composite function

$$y = g(f(x)).$$

$$\boxed{\frac{dy}{dx} = g'(f(x))\, f'(x)}$$

Chain Rule

$\frac{a}{b} = a b^{-1}$

Example $y = \sin(x^2)$

$f(x) = x^2$ $\qquad$ $g(x) = \sin x$

Sol^n

$$\frac{dy}{dx} = \cos(x^2)\, 2x$$

$$= 2x \cos(x^2).$$

Example $\quad y = (x^2 - x + 1)^7$

$f(x) = x^2 - x + 1 \qquad g(x) = x^7$

$$\frac{dy}{dx} = 7(x^2 - x + 1)^6 (2x - 1).$$

Example $\quad y = \sqrt{x^2 + 1}$

$$y = (x^2 + 1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} (2x) = \frac{2x}{2\sqrt{x^2 + 1}}$$

Quotient Rule

$$y = \frac{f(x)}{g(x)}$$

$$\frac{1}{x} = x^{-1}$$

$$y = f(x)\left(g(x)\right)^{-1}$$

$$\frac{dy}{dx} = f'(x)\left(g(x)\right)^{-1} + f(x)\left(\frac{d}{dx}\left(g(x)\right)^{-1}\right)$$

$$= f'(x)\left(g(x)\right)^{-1} - f(x)\left(g(x)\right)^{-2} g'(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{g(x)} - \frac{f(x)\,g'(x)}{g(x)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}}$$

QUOTIENT RULE