14. Inverse matrices

Consider the two matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}. \tag{14.1}$$

We can multiply them to find:

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I. \tag{14.2}$$

From equation 14.2 we deduce that the inverse of matrix A is

$$A^{-1} = B. \tag{14.3}$$

Two questions may come to mind: (i) how would we have found B had it not been given; (ii) does anybody really care about finding the inverse of A?

As a partial answer to Question (ii), consider the following system of *linear* equations.

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 1 \\ 2x & + & 5y & + & 5z & = & 2 \\ 3x & + & 8y & + & 6z & = & 3 \quad . \end{array} \tag{14.4}$$

These equations are said to be *linear* because the unkown quantities x, y, z don't appear as powers such as x^2, y^3, z^{-4} nor as products such as xy or xyz. This system of equations can be re-expressed as follows, using matrix multiplication.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{14.5}$$

To determine the values of x, y, z satisfying (14.4) we can multiply both sides of (14.5) by A^{-1} to obtain

$$AA^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{14.6}$$

and thus, from the above value of A^{-1} ,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{14.7}$$

We have illustrated how an inverse matrix can be used to find the solution $x = 1, y = 0, z = 0$ to a system of linear equations. Systems of linear equations abound in the sciences, social sciences, and engineering; opportunities for using inverse matrices also abound.

We now turn to Question (i): given a matrix A, how should we go about finding an inverse A^{-1}. We provide one of several answers to this.

14.1 Gauss-Jordan method for finding the inverse of an invertible matrix

Suppose that we wish to find the inverse of some $n \times n$ matrix A. The entries of A could be real numbers or they could be numbers in clock arithmetic. To cover both possibilities we let $\mathbb{K}$ denote the real numbers $\mathbb{R}$ or the integers $\mathbb{Z}_N$ modulo some positive integer N, and say that A is *over* $\mathbb{K}$. To find the inverse of A we can first use the $n \times n$ identity matrix I to form an $n \times 2n$ matrix $(A \mid I)$. We can then try to apply a sequence of suitably defined *elementary row operations*

$$(A \mid I) \xrightarrow{row-ops} (I \mid B) \tag{14.8}$$

to transform the $n \times 2n$-matrix to one of the form $(I \mid B)$ where B is some $n \times n$-matrix over $\mathbb{K}$. The definition of the elementary operations will ensure that $B = A^{-1}$. There are three allowable row operations:

(I) $R_i \leftarrow R_i + \lambda R_j \quad j \neq i, \lambda \in \mathbb{K}$.
Add a multiple of the jth row to the ith row.

(II) $R_i \leftarrow \lambda R_i \quad \lambda \in \mathbb{R}$, λ invrtible.
Multiply the ith row by an invertible number.

(III) $R_i \leftrightarrow R_j$.
Interchange the ith and jth rows.

To illustrate the method we use it to find the inverse of the above matrix A.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 3 & 8 & 6 & 0 & 0 & 1 \end{array}\right) \tag{14.9}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 2 & -3 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \leftarrow R_2 - 2R_1, \\ R_3 \leftarrow R_3 - 3R_1 \end{array} \tag{14.10}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right) \quad R_3 \leftarrow R_3 - 2R_2 \tag{14.11}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right) \quad R_3 \leftarrow -R_3 \tag{14.12}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -6 & 3 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right) \quad \begin{array}{l} R_1 \leftarrow R_1 - 3R_3 \\ R_2 \leftarrow R_2 + R_3 \end{array} \tag{14.13}$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -12 & 5 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right) \quad R_1 \leftarrow R_1 - 2R_2 \tag{14.14}$$

We conclude that

$$A^{-1} = \begin{pmatrix} 4 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}. \tag{14.15}$$

In the next lecture we'll explain why this process always furnishes the inverse of an invertible matrix.