13. Geometry of matrices

The linear transformation

$$T:\mathbb{R}^2\to\mathbb{R}^2,(x,y)\mapsto(3x+7y,2x+5y) \tag{13.1}$$

can be represented using matrix notation and the rules for matrix multiplication as follows.

$$T:\mathbb{R}^2\to\mathbb{R}^2,\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}3&7\\2&5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3x+7y\\2x+3y\end{pmatrix} \tag{13.2}$$

We say that the matrix

$$A=\begin{pmatrix}3&7\\2&5\end{pmatrix} \tag{13.3}$$

represents the linear transformation T.

In fact, the distributivity rule (9.5) for matrix multiplication

$$T:\mathbb{R}^2\to\mathbb{R}^2,\begin{pmatrix}x\\y\end{pmatrix}\mapsto A\begin{pmatrix}x\\y\end{pmatrix} \tag{13.4}$$

is a linear transformation. Thus *any* 2×2 matrix A represents a linear transformation of the plane in this way. The following result shows that the converse also holds.

Theorem 13.0.1 Any linear transformation of the plane

$$T:\mathbb{R}^2\to\mathbb{R}^2$$

can be represented by a 2×2 matrix

$$A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

of real numbers.

Proof. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an arbotrary linear transformation.

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \tag{13.5}$$

denote an arbitrary (column) vector in $\mathbb{R}^2$. Consider the two articular vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{13.6}$$

Now $T(e_1)$ and $T(e_2)$ have some values, say

$$T(e_1) = \begin{pmatrix} a \\ c \end{pmatrix}, \quad T(e_2) = \begin{pmatrix} b \\ d \end{pmatrix}. \tag{13.7}$$

Using (13.7) and the linearity of T we find:

$$\begin{aligned}
T(v) &= T\begin{pmatrix} x \\ y \end{pmatrix} && (13.8)\\
&= T(x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}) && (13.9)\\
&= T(xe_1 + ye_2) && (13.10)\\
&= xT(e_1) + yT(e_1) \qquad \text{(by the linearity of } T\text{)} && (13.11)\\
&= x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix} && (13.12)\\
&= \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} && (13.13)\\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. && (13.14)
\end{aligned}$$

Hence T is represented by a 2×2 matrix of real numbers. ■

■ **Example 13.1** A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $T(1,2) = (1,3)$ and $T(3,4) = (4,5)$. Find a formula for $T(x,y)$.

We have:

$$\begin{aligned}
T(1,0) &= T(2(1,2) - (3,4)) && (13.15)\\
&= 2T(1,2) - T(3,4) && (13.16)\\
&= (2,6) - (4,5) && (13.17)\\
&= (-2,1) && (13.18)
\end{aligned}$$

$$\begin{aligned}
T(0,1) &= T(-\frac{3}{2}(1,2) + \frac{1}{2}(3,4)) && (13.19)\\
&= -\frac{3}{2}T(1,2) + \frac{1}{2}T(3,4) && (13.20)\\
&= -\frac{3}{2}(1,3) + \frac{1}{2}(4,5) && (13.21)\\
&= (\frac{1}{2}, -2) && (13.22)
\end{aligned}$$

Hence

$$T(x,y) = x(-2,1) + y(\frac{1}{2},-2) = (-2x + \frac{1}{2}y, x - 2y) \tag{13.23}$$

is the required formula. ■

13.1 Why did that proof work?

Having read the proof of a theorem line by line, making sure that each line follows from previous lines, we are in a position to say "Ah, now I know for sure that the theorem is true". But a line by line reading may not be enough for us to say "Ah, now I really see why the result is true". To "really see" why a result is true, we might need to develop an informative overview of the proof.

The proof of Theorem 13.0.1 has two main ingredients: i) linearity of T is fundamental to the argument; ii) the choice of the two vectors e_1 and e_2 is the other key ingredient. The two vectors e_1 and e_2 have the property that for any vector $v \in \mathbb{R}^2$ there is a unique pair of real numbers x, y such that $v = xe_1 + ye_2$. In fact, this is the only property of e_1 and e_2 that was used. So in the proof we could have taken *any* two vectors with this property. A pair of vectors with this property is called a *basis* of $\mathbb{R}^2$. From the geometric picture for addition of vectors presented in the previous lecture, it is clear that any pair of vectors will do providing that the rays from the origin to each of them them are not colinear. The particular pair e_1, e_2 used in the proof is said to be the *standard basis* of $\mathbb{R}^2$.

Now that we can see why Theorem 13.0.1 is true, it is routine to generalize its ingredients to the following definition and result about $n \times n$ matrices.

Definition 13.1.1 A transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *linear* if
1. $T(P+Q) = T(P) + T(Q)$
2. $T(\lambda P) = \lambda T(P)$

for all $P, Q \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

Theorem 13.1.1 Any linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

can be represented as

$$T : \mathbb{R}^n \to \mathbb{R}^n, v \mapsto Av$$

where A is an $n \times n$ matrix.

The proof of Theorem 13.1.1 uses the following definition.

Definition 13.1.2 A list of n vectors $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ is said to be a basis for $\mathbb{R}^n$ if, for any vector $v \in \mathbb{R}^n$, there is a unique list of real numbers $x_1, x_2, \dots, x_n \in \mathbb{R}$ such that

$$v = x_1e_1 + x_2e_2 + \cdots + x_ne_n \ .$$

■ **Example 13.2** The vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{13.24}$$

form a basis for $\mathbb{R}^3$. This particular basis is called the *standard basis* for $\mathbb{R}^3$. ■

13.2 Matrix multiplication explained

On first encounter the formula (9.11) for matrix multiplication may seem puzzling. It may seem like pure chance, or perhaps clever ingenuity on the part of its inventor, that this multiplication satisfies familiar properties such as associativity (9.5) and distibutivity over addition (9.5). However, the following theorem shows that formula (9.11) is immediately stumbled upon by anyone wishing to calculate with composites of linear transformations.

Theorem 13.2.1 Let $S\colon \mathbb{R}^n \to R^n, v \mapsto Av$ and $T\colon \mathbb{R}^n \to R^n, v \mapsto Bv$ be two linear transformations represented by $n \times n$ matrices A and B respectively. Then the composite function

$$S \circ T\colon \mathbb{R}^n \to R^n, v \mapsto S(T(v)) \tag{13.25}$$

satisfies

$$S \circ T(v) = (AB)v\ . \tag{13.26}$$

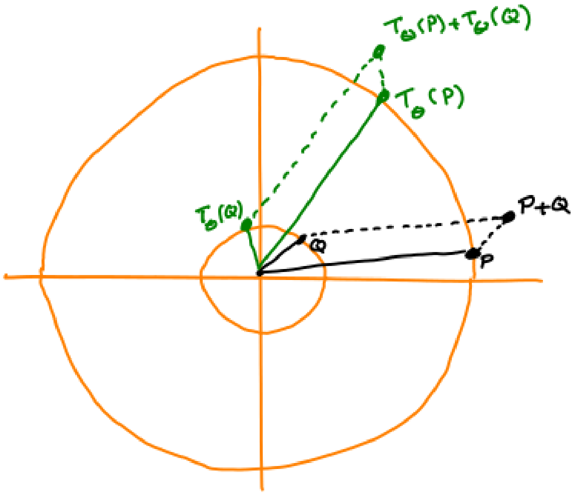
To prove this theorem one just needs to verify the equality (13.26). It is a straightforward and worthwile excercise to verify (13.26) for $n = 2, 3$, after which it will be clear that the equality holds for all $n \geq 1$.

Since the product AB of two $n \times n$ matrices is itself an $n \times n$ matrix, and since multiplication by a matrix represents a linear transformation, we arrive at the following important consequence of Theorem 13.2.1.

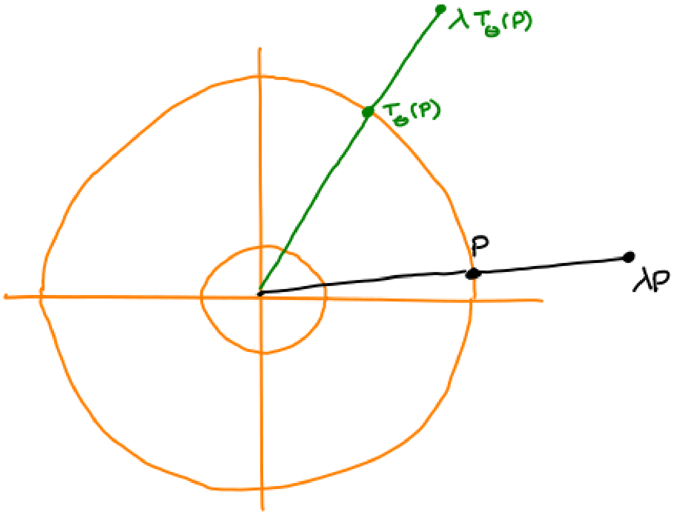
Corollary 13.2.2 The composite $S \circ T$ of two linear transformations of $\mathbb{R}^n$ is itself a linear transformation.

13.3 Rotations and reflections

Let $T_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the transformation that rotates the real plane anticlockwise through an angle θ about the origin $(0,0)$. The following diagram pictures two parallelograms, one rotated clockwise through an angle θ. One parallelogram represents the addition of two points P, Q. The other parallelogram represents the addition of $T_\theta(P)$, $T_\theta(Q)$.



From this diagram we see that $T_\theta(P+Q) = T_\theta(P) + T_\theta(Q)$. The next diagram pictures a point P, a scalar multiple of it λP, the point $T_\theta(P)$ and the scalar multiple $\lambda T_\theta(P)$.



We see that $T_\theta(\lambda P) = \lambda T_\theta(P)$. So we have proved the following.

Theorem 13.3.1 A rotation $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane about the origin is a linear transformation.

In Example 12.3 we proved that reflection in the line $y = x$ is a linear transformation of the plane. If we take an arbitrary line through the origin then reflection in this arbitrary line is the same as an anticlockwise rotation about the origin through an angle θ so that the arbitrary line coincides with the line $y = x$, followed by reflection in the line $y = x$, followed by a clockwise rotation through θ. The following theorem follows from Corollary 13.2.2.

Theorem 13.3.2 A reflection $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the plane in a line through the origin is a linear transformation.

■ **Example 13.3** Find a formula for the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ consisting of reflection in the y-axis followed by clockwise rotation about the origin through an angle of $5\pi/2$ radians.

To find a formula we first note that the reflection and rotation are both linear transformations. Hence their composite T is a linear transformation. A formula can thus be derived from the values of $T(1,0)$ and $T(0,1)$. By inspection, $T(1,0) = (0,1)$ and $T(0,1) = (1,0)$. The required formula is thus:

$$T(x,y) = (y,x) \tag{13.27}$$

Note that this formula tells us that the composite transformation T is simply the transformation that reflects in the line $y = x$. ■