12. Linear transformations of the plane

Who invented the rules for matrix multiplication? Why were they invented? The second of these questions is the easier, and will be addressed next lecture after some necessary background has been covered. The mathematics literature shows that several people *discovered* the rules involved in matrix multiplication – they arise naturally as the answer to various fundamental problems about areas, volumes, and geometric transformations. The English mathematician James Joseph Sylvester is credited with having coined the term *matrix* (which is the Latin for *womb*) in a paper of 1848. The English mathematician Arthur Cayley nurtured the theory of matrices and included the definition of matrix multiplication – he used the term *matrix composition*– in a paper of 1858, though that paper only considered 2×2 and 3×3 matrices.

The theory of matrices was developed in two quite different ways by its early pioneers. Sylvester and Cayley are examples of mathematicians who emphasized abstract algebraic structure. The Irish mathematician William Rowan Hamilton and German mathematician Herman Grassmann are examples of mathematicians who favoured a geometric view of matrices. The theory has benefited enormously from the complementarity of these two approaches.

12.1 Algebra versus Geometry

The preceding lectures have focused on algebraic aspects of matrices – their addition, multiplication, division and the relationships satisfied by these operations. Our next aim is to provide a geometric view of matrices. The following excerpt from a lecture delivered by Fields Medallist Michael Atiyah in 2000 explains the case for geometry.

> Let me try to explain my own view of the difference between geometry and algebra. Geometry is, of course, about space, of that there is no question. If I look out at the audience in this room I can see a lot; in one single second or microsecond I can take in a vast amount of information, and that is of course not an accident. Our brains have been constructed in such a way that they are extremely concerned with vision. Vision, I understand from friends who work in neurophysiology, uses up something like 80 or 90 percent of the cortex of the brain. There are about 17 different

centres in the brain, each of which is specialised in a different part of the process of vision: some parts are concerned with vertical, some parts with horizontal, some parts with colour, or perspective, and finally some parts are concerned with meaning and interpretation. Understanding, and making sense of, the world that we see is a very important part of our evolution. Therefore, spatial intuition or spatial perception is an enormously powerful tool, and that is why geometry is actually such a powerful part of mathematics—not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool. That is quite clear if you try to explain a piece of mathematics to a student or a colleague. You have a long difficult argument, and finally the student understands. What does the student say? The student says, 'I see!' Seeing is synonymous with understanding, and we use the word 'perception' to mean both things as well. At least this is true of the English language. It would be interesting to compare this with other languages. I think it is very fundamental that the human mind has evolved with this enormous capacity to absorb a vast amount of information, by instantaneous visual action, and mathematics takes that and perfects it.

Algebra, on the other hand (and you may not have thought about it like this), is concerned essentially with time. Whatever kind of algebra you are doing, a sequence of operations is performed one after the other and 'one after the other' means you have got to have time. In a static universe you cannot imagine algebra, but geometry is essentially static. I can just sit here and see, and nothing may change, but I can still see. Algebra, however, is concerned with time, because you have operations which are performed sequentially and, when I say 'algebra', I do not just mean modern algebra. Any algorithm, any process for calculation, is a sequence of steps performed one after the other; the modern computer makes that quite clear. The modern computer takes its information in a stream of zeros and ones, and it gives the answer.

Algebra is concerned with manipulation in time and geometry is concerned with space. These are two orthogonal aspects of the world, and they represent two different points of view in mathematics. Thus the argument or dialogue between mathematicians in the past about the relative importance of geometry and algebra represents something very, very fundamental.

Of course it does not pay to think of this as an argument in which one side loses and the other side wins. I like to think of this in the form of an analogy: 'Should you just be an algebraist or a geometer?' is like saying 'Would you rather be deaf or blind?' If you are blind, you do not see space: if you are deaf, you do not hear, and hearing takes place in time. On the whole, we prefer to have both faculties.

In physics, there is an analogous, roughly parallel, division between the concepts of physics and the experiments. Physics has two parts to it: theory—concepts, ideas, words, laws—and experimental apparatus. I think that concepts are in some broad sense geometrical, since they are concerned with things taking place in the real world. An experiment, on the other hand, is more like an algebraic computation. You do something in time; you measure some numbers; you insert them into formulae, but the basic concepts behind the experiments are a part of the geometrical tradition.

One way to put the dichotomy in a more philosophical or literary framework is to say that algebra is to the geometer what you might call the 'Faustian offer'. As you know, Faust in Goethe's story was offered whatever he wanted (in his case the love of a beautiful woman), by the devil, in return for selling his soul. Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.' (Nowadays you can think of it as a computer!) Of course we like to have things both ways; we would probably cheat on the devil, pretend we are selling our soul, and not give it away. Nevertheless, the danger to our soul is there, because when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning.

I am a bit hard on the algebraists here, but fundamentally the purpose of algebra always was to produce a formula which one could put into a machine, turn a handle and get the answer. You took something that had a meaning; you converted it into a formula, and you got out the answer. In that process you do not need to think any more about what the different stages in the algebra correspond to in the geometry. You lose the insights, and this can be important at different stages. You must not give up the insight altogether! You might want to come back to it later on. That is what I mean by the Faustian offer. I am sure it is provocative.

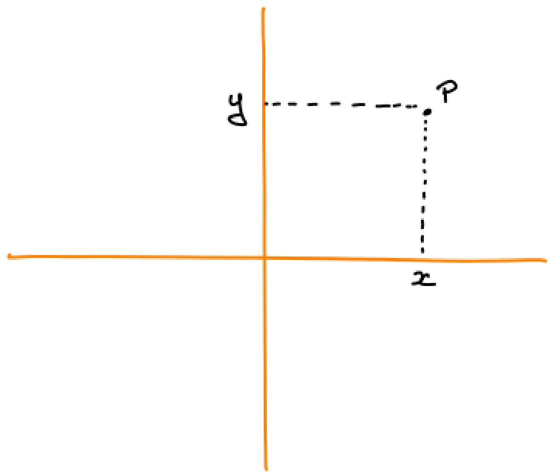
This choice between geometry and algebra has led to hybrids which confuse the two, and the division between algebra and geometry is not as straightforward and naïve as I just said. For example, algebraists frequently will use diagrams. What is a diagram except a concession to geometrical intuition?

12.2 Linear transformations

We refer to the set $\mathbb{R}$ of real numbers as the *real line* and picture it as an infinite line with no start or end points. We refer to the set

$$\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\} \tag{12.1}$$

of pairs of real numbers as the *real plane*, or just *plane*, and picture it as an infinite plane with no boundary.



Any point P in the plane can be represented by a pair (x,y) of real numbers.

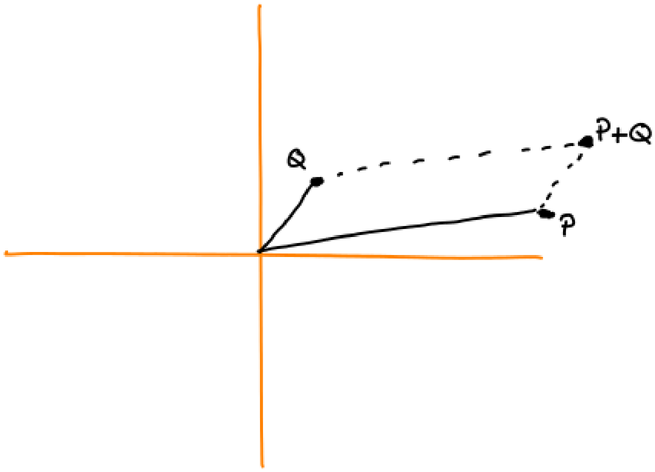
A *transformation* of the plane is just a function

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \tag{12.2}$$

which sends each point $P = (x,y)$ to some point $T(P)$ in the plane.

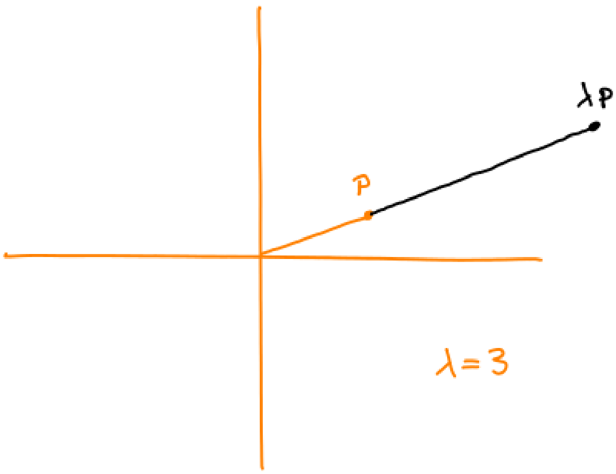
We can add two points $P = (x,y)$ and $Q = (x',y')$ using matrix addition of row vectors.

$P+Q=(x+x',y+y')$



We can multiply a point $P=(x,y)$ by a scalar $\lambda \in \mathbb{R}$ using scalar multiplication of matrices.

$\lambda P=(\lambda x,\lambda y)$



Definition 12.2.1 A transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ is said to be *linear* if

1. $T(P+Q)=T(P)+T(Q)$
2. $T(\lambda P)=\lambda T(P)$

for all $P,Q \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$.

■ **Example 12.1** Consider the transformation

$$T:\mathbb{R}^2 \to \mathbb{R}^2,(x,y)\mapsto(3x+7y,2x+5y)\ . \tag{12.3}$$

So, for instance, $T(-3,1)=(-2,-1)$.

For arbitrary points $P = (x,y)$, $Q = x',y')$ we find:

$$T(P+Q) = T(x+x', y+y') \tag{12.4}$$
$$= (3(x+x')+7(y+y'), 2(x+x')+5(y+y')) \tag{12.5}$$
$$= (3x+7y+3x'+7y', 2x+5y+2x'+5y') \tag{12.6}$$
$$= (3x+7y, 2x+5y)+(3x'+7y', 2x'+5y') \tag{12.7}$$
$$= T(P)+T(Q) \tag{12.8}$$

Also, for any $\lambda \in \mathbb{R}$ we find:

$$T(\lambda P) = T(\lambda x, \lambda y) \tag{12.9}$$
$$= (3\lambda x+7\lambda y, 2\lambda x+5\lambda y)) \tag{12.10}$$
$$+\lambda(3x+7y, 2x+5y) \tag{12.11}$$
$$= \lambda T(P) \tag{12.12}$$

Thus T is a linear transformation. ■

■ **Example 12.2** Consider the transformation

$$T: \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto (x^2, y^2)\ . \tag{12.13}$$

So, for instance, $T(-3,1) = (9,1)$.

For $P = (1,2)$ and $\lambda = 3$ we find:

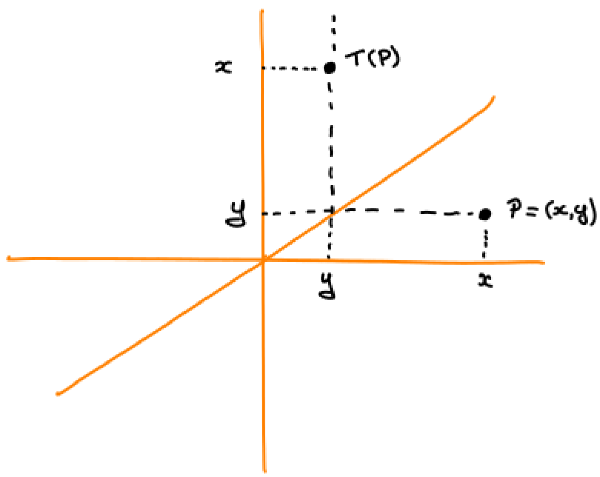
$$T(\lambda P) = T(3,6) \tag{12.14}$$
$$= (9,36) \tag{12.15}$$

$$\lambda T(P) = 3T(1,2) \tag{12.16}$$
$$= 3(1,4) \tag{12.17}$$
$$= (3,12) \tag{12.18}$$

Since $T(\lambda P) \neq \lambda T(P)$ in this particular case, we conclude that T is not a linear transformation. ■

■ **Example 12.3** Let : $\mathbb{R}^2 \to \mathbb{R}^2$ be the transformation of the plane obtained by reflecting in the line $y = x$. To decide whether T is linear we need to find an algebraic formula for the transformation.

From the diagram



we find:

$$T(x,y) = (y,x) \tag{12.19}$$

For arbitrary points $P = (x,y), Q = (x',y')$ and scalar $\lambda \in \mathbb{R}$ we find:

$$T(P+Q) = T(x+x',y+y') \tag{12.20}$$
$$= (y+y',x+x') \tag{12.21}$$
$$= (y,x)+(y',x') \tag{12.22}$$
$$T(P)+T(Q) \tag{12.23}$$

$$T(\lambda P) = T(\lambda x, \lambda y) \tag{12.24}$$
$$= (\lambda y, \lambda x) \tag{12.25}$$
$$= \lambda(y,x) \tag{12.26}$$
$$\lambda T(P) \tag{12.27}$$

Hence reflection in the line $y = x$ is a linear transformation. ■

12.3 Transformations that preserve lines

Let P and V be row vectors in $\mathbb{R}^2$. Any set of the form

$$L = \{P + \lambda V \ : \ \lambda \in \mathbb{R}\} \tag{12.28}$$

is called a *line*. We think of L as the collection of points in the plane that 'can be reached by starting at P and travelling some distance in the direction of V'. Note that we allow the case $V = 0$ in which the line L consists of just a single point.

A transformation of the plane $T : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be *line preserving* if the set

$$T(L) = \{T(v) : \ v \in L\} \tag{12.29}$$

is a line whenever L is a line. Since a linear transformation T satisfies $T(P+\lambda V) = T(P)+\lambda T(V)$ we immediately have the following result.

Proposition 12.3.1 Any linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is line preserving.

So linear transformations do indeed have something to do with lines. However, not every line preserving transformation is linear!