

## 9. Basic arithmetic

Clock arithmetic is very much like the arithmetic taught in school. In particular, the following rules

$$\begin{aligned}a + b &= b + a && \text{(commutative addition)} \\ab &= ba && \text{(commutative multiplication)} \\(a + b) + c &= a + (b + c) && \text{(associative addition)} \\(ab)c &= a(bc) && \text{(associative multiplication)} \\a(b + c) &= ab + ac && \text{(distributivity)}\end{aligned}$$

hold for any  $a, b, c \in \mathbb{Z}_m$  and integer  $m \geq 1$ . We'll now study an arithmetic where multiplication is not necessarily commutativity.

### 9.1 Matrices

A *matrix* is an array of numbers arranged neatly in rows and columns. The rows all have the same length. The columns all have the same length. Some examples are

$$\begin{pmatrix} 1 & 2 & 5 \\ -2 & 3 & 10 \end{pmatrix} \quad 2 \times 3 \text{ matrix,} \quad (9.1)$$

$$\begin{pmatrix} \frac{1}{2} & -\sqrt{2} \\ 3 & 7 \end{pmatrix} \quad 2 \times 2 \text{ matrix,} \quad (9.2)$$

$$(1 \ 2 \ 3 \ -4 \ 5) \quad 1 \times 5 \text{ matrix, also called a } \textit{row vector}, \quad (9.3)$$

$$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad 3 \times 1 \text{ matrix, also called a } \textit{column vector}. \quad (9.4)$$

### 9.2 Matrix addition

Two  $m \times n$  matrices  $A, B$  are *added* by adding corresponding entries.

■ **Example 9.1**

$$\begin{array}{rcl}
 & A & B & = & A+B \\
 \left( \begin{array}{ccc} 17 & 22 & 42 \\ 6 & 18 & 5 \end{array} \right) & + & \left( \begin{array}{ccc} 1 & 2 & 3 \\ -6 & 1 & 5 \end{array} \right) & = & \left( \begin{array}{ccc} 18 & 24 & 45 \\ 0 & 19 & 10 \end{array} \right) \\
 \left( \begin{array}{cc} -2 & 3 \\ 1 & 4 \end{array} \right) & + & \left( \begin{array}{cc} 7 & 6 \\ 5 & 4 \end{array} \right) & = & \left( \begin{array}{cc} 5 & 9 \\ 6 & 8 \end{array} \right) \\
 (-2 & 3 & 1 & 0) & + & (7 & 6 & -1 & -2) & = & (5 & 9 & 0 & -2) \\
 \left( \begin{array}{ccc} 17 & 22 & 42 \\ 6 & 18 & 5 \end{array} \right) & + & \left( \begin{array}{cc} 7 & 6 \\ 5 & 4 \end{array} \right) & & & \text{not defined}
 \end{array}$$

Given a matrix  $A$  we write  $-A$  to denote the matrix obtained by placing a minus in front of each entry of  $A$ .

■ **Example 9.2**

$$\begin{array}{ccc}
 & A & -A \\
 \left( \begin{array}{cc} 2 & -3 \\ 4 & 7 \end{array} \right) & & \left( \begin{array}{cc} -2 & 3 \\ -4 & -7 \end{array} \right)
 \end{array}$$

For any  $m \times n$  matrix  $A$  the sum  $A + (-A)$  yields the  $m \times n$  matrix whose entries are all zero. Such a matrix is called a *null matrix* or *zero matrix*. We write

$$A + (-A) = 0 \tag{9.5}$$

where here  $0$  denotes a null matrix.

### 9.3 Multiplication of a column vector by row vector

Let

$$R = (a_1, a_2, \dots, a_n) \tag{9.6}$$

be a row vector of length  $n$ . Let

$$C = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{9.7}$$

be a column vector of length  $n$ . The *product*  $R.C$  is a number defined as:

$$R.C = (a_1, a_2, \dots, a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n \tag{9.8}$$

■ **Example 9.3** For example:

$$R.C = (-2, 3, 7) \begin{pmatrix} 8 \\ 5 \\ 9 \end{pmatrix} = -2 \times 8 + 3 \times 5 + 7 \times 9 = 62 \tag{9.9}$$

If we regard the number  $R.C$  as a  $1 \times 1$  matrix then we obtain the 'formula':

$$(1 \times n \text{ matrix}) \times (n \times 1 \text{ matrix}) = (1 \times 1 \text{ matrix}) \quad (9.10)$$

We'll see below that it is also possible to define the product  $C.R$ , though in this case our definition will yield an  $n \times n$  matrix. In particular, we'll get  $R.C \neq C.R$  whenever  $n > 1$ .

## 9.4 Matrix multiplication

Let  $A$  be an  $m \times n$  matrix. We can view  $A$  as a list of  $m$  row vectors. We write  $R_i^A$  to denote the  $i$ th row vector of  $A$ .

Let  $B$  be an  $n \times p$  matrix. We can view  $B$  as a list of  $p$  column vectors. We write  $C_j^B$  to denote the  $j$ th column vector of  $B$ .

For each  $1 \leq i \leq m$  and  $1 \leq j \leq p$  we can multiply the  $i$ th row of  $A$  and  $j$ th column of  $B$  to obtain the number  $R_i^A C_j^B$ . These numbers are used to define the matrix product  $AB$  as:

$$AB = \begin{pmatrix} R_1^A C_1^B & R_1^A C_2^B & R_1^A C_3^B & \cdots & R_1^A C_p^B \\ R_2^A C_1^B & R_2^A C_2^B & R_2^A C_3^B & \cdots & R_2^A C_p^B \\ R_3^A C_1^B & R_3^A C_2^B & R_3^A C_3^B & \cdots & R_3^A C_p^B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_m^A C_1^B & R_m^A C_2^B & R_m^A C_3^B & \cdots & R_m^A C_p^B \end{pmatrix} \quad (9.11)$$

Thus  $AB$  is an  $m \times p$  matrix whose entry in the  $i$ th row and  $j$ th column is the product of the  $i$ th row of  $A$  and  $j$ th column of  $B$ . We have the 'formula':

$$(m \times n \text{ matrix}) \times (n \times p \text{ matrix}) = (m \times p \text{ matrix}) \quad (9.12)$$

■ **Example 9.4** As an example of this definition of matrix multiplication we have:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 3 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 14 & 7 \\ 1 & 35 & 19 \end{pmatrix} \quad (9.13)$$

■

## 9.5 Algebraic properties

The equalities

$$\begin{aligned} A + B &= B + A && \text{(commutative addition)} \\ (A + B) + C &= A + (B + C) && \text{(associative addition)} \end{aligned}$$

hold for any three  $m \times n$  matrices  $A, B, C$ .

The equalities

$$\begin{aligned} (AB)C &= A(BC) && \text{(associative multiplication)} \\ A(B + B') &= AB + AB' && \text{(distributivity)} \end{aligned}$$

hold for any  $m \times n$  matrix  $A$ ,  $n \times p$  matrices  $B, B'$ , and  $p \times q$  matrix  $C$ .

