Left-hand and right-hand limits

We write

$$\lim_{x \to a-} f(x) = l$$

to mean that $f(x)$ is close to l for all x sufficiently close to a and strictly less than a

$0 < a - x < \delta$ left-hand

$0 < x - a < \delta$ right-hand

Proposition

$$\lim_{x \to a} f(x) = l$$

if and only if

$$\lim_{x \to a-} f(x) = l = \lim_{x \to a+} f(x)$$

Example Let

$$f(x) = \begin{cases} x^2+1 & \text{if } x < 3 \\ x+c & \text{if } x \geq 3 \end{cases}$$

where c is some constant.

For what value of c does

$$\lim_{x \to 3} f(x)$$

exist?

Sol^n

$$\lim_{x \to 3^-} f(x) = 10$$

$$\lim_{x \to 3^+} f(x) = 3 + c$$

Need

$$10 = 3 + c.$$

So $\lim_{x \to 3} f(x)$ exists if and only if $c = 7$.

In this case $\lim_{x \to 3} = 10$.

Limits at infinity

Defn

$$\lim_{x \to \infty} f(x) = \lim_{x \to 0+} f\left(\frac{1}{x}\right)$$

Defn

$$\lim_{x \to -\infty} f(x) = \lim_{x \to 0-} f\left(\frac{1}{x}\right)$$

$$(A+B)(A-B) = A^2 - B^2$$

Example Evaluate

$$l = \lim_{x \to \infty} \frac{1}{\sqrt{x^2+2x} - x}$$

Soln

$$l = \lim_{x \to \infty} \frac{1}{\sqrt{x^2+2x} - x} \cdot \frac{(\sqrt{x^2+2x} + x)}{(\sqrt{x^2+2x} + x)}$$

$$\begin{matrix} A & - & B & & A & + & B \end{matrix}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x^2+2x} + x}{x^2+2x - x^2} = \lim_{x \to \infty} \frac{\sqrt{x^2+2x} + x}{2x}$$

$$\frac{\sqrt{8}}{2} = \sqrt{\frac{8}{4}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{\frac{x^2+2x}{x^2}} + 1}{2} = \lim_{x \to \infty} \frac{\sqrt{1 + \frac{2}{x}} + 1}{2}$$

$$= \frac{1+1}{2} = 1$$

Example What are the horizontal and vertical asymptotes

$$y = \frac{2x-5}{3x+2} \; ?$$

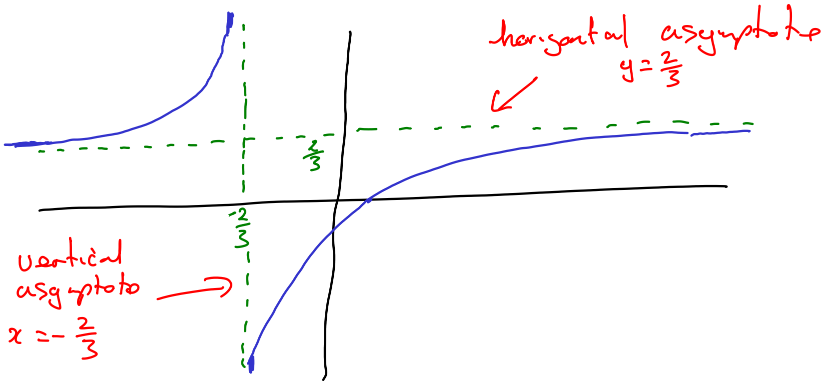
Domain of $y = f(x)$ is $\mathbb{R} \setminus \{-\frac{2}{3}\}$

Sketch the graph of y.

Sol^n

$$\lim_{x \to \infty} \frac{2x-5}{3x+2} = \frac{2}{3}$$

$$\lim_{x \to -\infty} \frac{2x-5}{3x+2} = \frac{2}{3}$$



Example $\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$

Sol^n

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{x^2} \frac{(1 + \cos x)}{(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x}{x^2 (1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2 (1 + \cos x)}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \frac{1}{(1 + \cos x)}$$

$$= \left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2 \lim_{x \to 0} \left(\frac{1}{1 + \cos x}\right)$$

$$= \lim_{x \to 0} \frac{1}{1 + \cos x} = \frac{1}{2}.$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\sin^2 x = (\sin x)(\sin x)$$

$$\cos(0) = 1$$