Second homework due : Friday 30 October

Intermediate Value Theorem

Suppose

$$y = f(x)$$

is a real valued function which is continuous at all points x in the interval $a \le x \le b$.

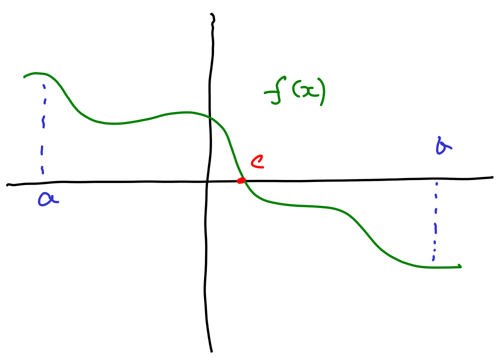
Suppose also that

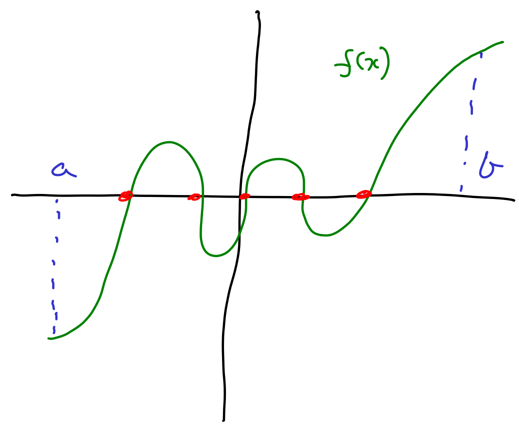
$$f(a)\,f(b) \le 0 .$$

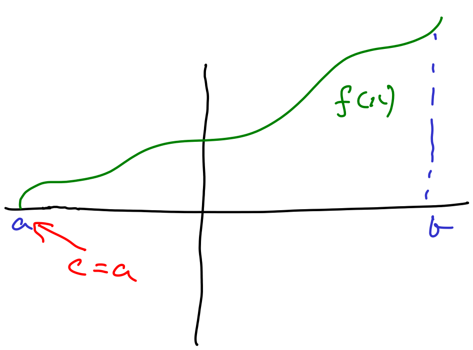
Then there exists at least one number c in the interval $a \le c \le b$ such that

$$f(c) = 0 .$$

axiom: statement which we accept to be true without proof







<u>Example</u> Prove that

$$x^3 - x - 1 = 0 \qquad (*)$$

has a solution in the range $1 \leq x \leq 2$.

<u>proof</u> Let $f(x) = x^3 - x - 1$.

"Clearly" $f(x)$ is continuous.

Let $a = 1$, $b = 2$.

$$\left.\begin{array}{l} f(1) < 0 \\ f(2) > 0 \end{array}\right\} \text{ Thus } f(1)\,f(2) \leq 0.$$

The IVT tells us that there exists at least one c in the range $1 \leq c \leq 2$ such that $f(c) = 0$. Hence equation $(*)$ has a solution in the interval $[1, 2]$.

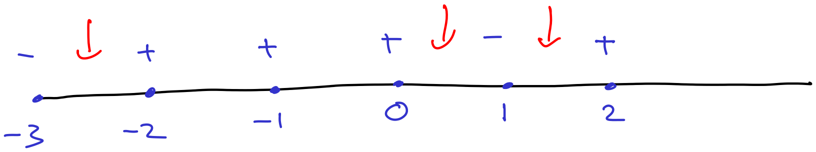
Example Prove that

$$x^3 - 4x + 1 = 0$$

has exactly three real solutions, and find approximations to them.

Sol^n Let $f(x) = x^3 - 4x + 1$.

Note that f is continuous on $\mathbb{R}$.



$f(0) > 0$ $\quad$ $f(2) > 0$

$f(1) < 0$ $\quad$ $f(-2) > 0$

$f(-1) > 0$ $\quad$ $f(-3) < 0$

The IVT says that $f(c) = 0$ for

some $c \in [1, 2]$

and

some $c \in [0, 1]$

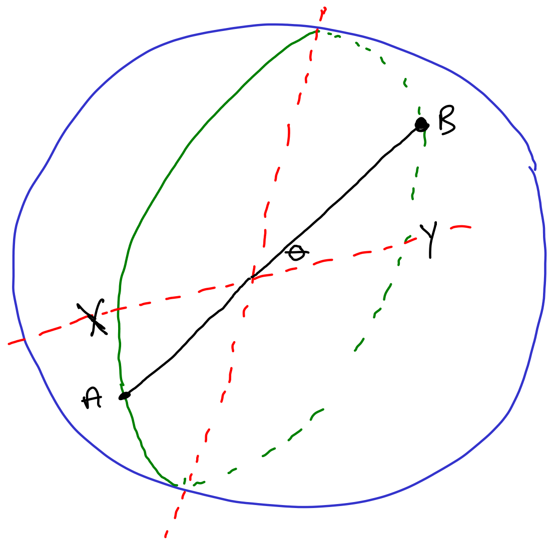
and

some $c \in [-3, -2]$.

Since $f(x)$ has degree 3 "we know" that f can have at most three zeros.

Hence f has exactly three zeros.

Application of IVT



Take a great circle on the earth.

FACT There exist opposite points on your chosen great circle with equal air pressure.

Explanation

Consider

$f(\theta)$ = air pressure at A − air pressure at B

Note: $f(\theta)$ is a continuous function of θ.

I want to prove that, for some θ,

pressure at A = pressure at B.

i.e. I want to prove that, for some $\theta \in [0, \pi]$,

$$f(\theta) = 0.$$

If $f(0) = 0$, or if $f(\pi) = 0$ then we'd have $f(0) f(\pi) \le 0$.

Suppose $f(0) \ne 0$ and $f(\pi) \ne 0$.

Note: $f(0) f(\pi) < 0$

$f(0)$ = air pressure at X − air pressure at Y

$f(\pi)$ = air pressure at Y − air pressure at X

So the IVT says that there is at least one $\theta \in [0, \pi]$ with $f(\theta) = 0$.

QED.