$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$

Theorem
Proposition
Lemma
} true statement for which there is a proof

Conjecture } statement for which there is currently no proof

Sandwich Lemma

Suppose

$$f(x) \leq g(x) \leq h(x)$$

for all x sufficiently near $a \in \mathbb{R}$ (except possible for $x = a$). Suppose also

$$\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x).$$

Then

$$\lim_{x \to a} g(x) = l$$

Exercise: try to prove it.

Example Evaluate $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Sol^n

$$g(x) = x^2 \sin\left(\tfrac{1}{x}\right)$$

$$f(x) = -x^2$$

$$h(x) = x^2$$

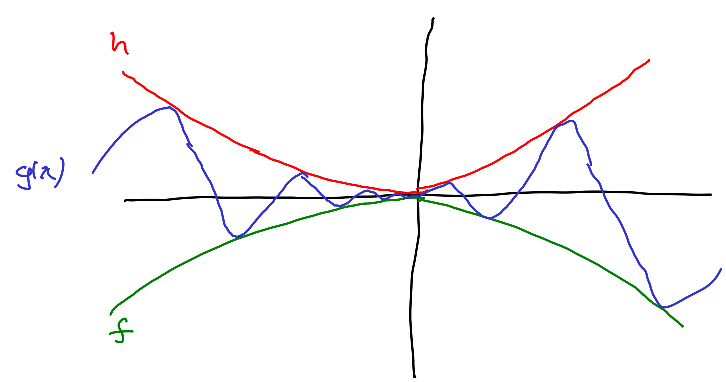
Domain$(g) = \mathbb{R} \setminus \{0\}$
Range$(g) = \mathbb{R}$

For x near and $x \neq 0$ we have

$$-x^2 \leq x^2 \sin\left(\tfrac{1}{x}\right) \leq x^2$$

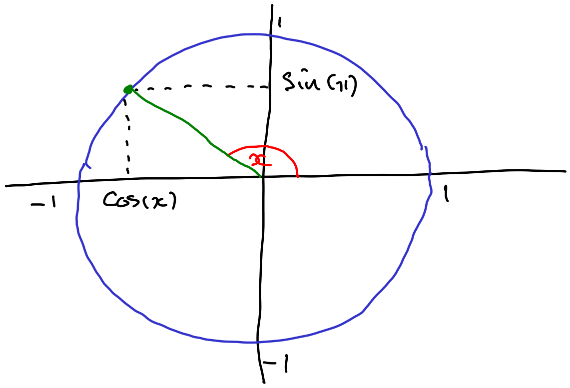
$$\lim_{x \to 0} -x^2 = 0$$

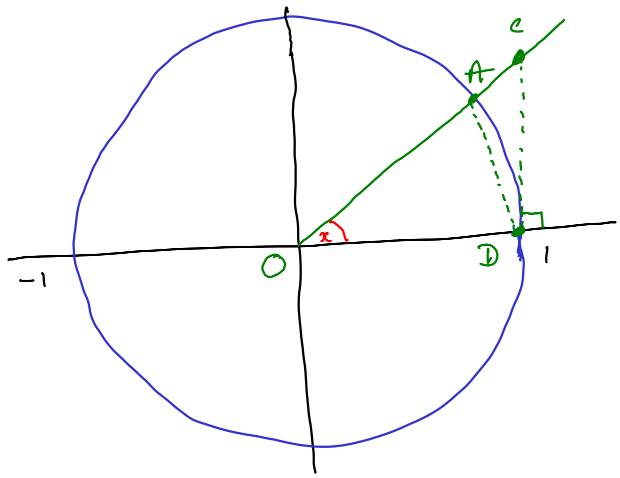
$$\lim_{x \to 0} x^2 = 0$$

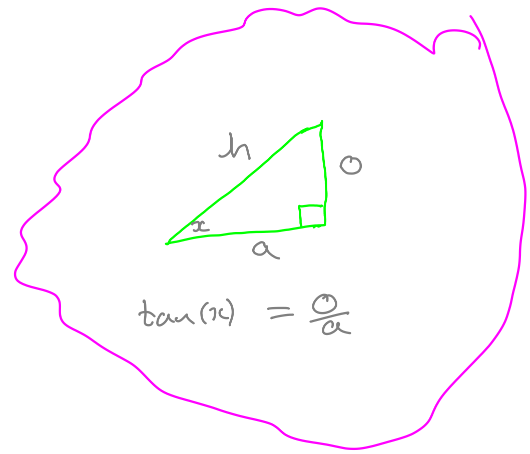


Thus, by the Sandwich Lemma

$$\lim_{x \to 0} x^2 \sin\left(\tfrac{1}{x}\right) = 0.$$







Note:

area of triangle OAD $\leq$ area of sector OAD $\leq$ area of triangle OCD

so

$$\frac{1}{2}\sin x \leq \frac{x}{2} \leq \frac{1}{2}\tan x$$

or

$$\sin x \leq x \leq \frac{\sin x}{\cos x}$$

So

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

So, for small $x > 0$

$$1 \geq \frac{\sin x}{x} \geq \cos x$$

Now, as $x \to 0$ we have $\cos x \to 1$.

"Hence," from the Sandwich Lemma, we have

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$