We write

$$\lim_{x \to 1} g(x) = 6$$

to mean that

for any $\varepsilon > 0$ there exists a $\delta > 0$

such that

$$0 < |x-1| < \delta$$

implies

$$|g(x) - 6| < \varepsilon .$$

Proposition

$$\lim_{x \to 1} \frac{x^6 - 1}{x - 1} = 6.$$

Proof Let $\varepsilon > 0$. We can choose

$$\delta = \min\left(1, \frac{\varepsilon}{5 \times 31}\right).$$

If $|x-1| < \delta$ then

$|1-x| < \frac{\varepsilon}{5}$

$|1-x^2| = |1-x||1+x| < \frac{\varepsilon}{5\times 31}|1+x| < \frac{\varepsilon}{5\times 31}\times 3 < \frac{\varepsilon}{5}$

$|1-x^3| = |1-x||1+x+x^2| < \frac{\varepsilon}{5\times 31}|1+x+x^2| < \frac{\varepsilon}{5\times 31}\times 7 < \frac{\varepsilon}{5}$

$|1-x^4| = \qquad < \frac{\varepsilon}{5}$

$|1-x^5| = \qquad < \frac{\varepsilon}{5}.$

If $0 < |1-x| < \delta$ then

$$\left|\frac{x^6-1}{x-1} - 6\right| = \left|\frac{(x-1)(1+x+x^2+x^3+x^4+x^5)}{(x-1)} - 6\right|$$

$= |1+x+x^2+x^3+x^4+x^5 - 6|$

$= |(1-1)+(x-1)+(x^2-1)+(x^3-1)+(x^4-1)+(x^5-1)|$

$\leq |1-1| + |x-1| + |x^2-1| + |x^3-1| + |x^4-1| + |x^5-1|$

$< 0 + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5}$

$= \varepsilon.$

QED

$|a+b| \leq |a| + |b|$

$|3+(-2)| \leq |3| + |-2| = 3+2 = 5$

$\|$

$|1| = 1$

Proposition Suppose

$\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$

both exist. Then

i) $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

ii) for any $k \in \mathbb{R}$

$\lim_{x \to a} k f(x) = k \lim_{x \to a} f(x)$.

iii) $\lim_{x \to a} (f(x) g(x)) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$.

iv) $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

provided $\lim_{x \to a} g(x) \neq 0$.

$\lim_{x \to 0} \frac{1}{x}$ does not exist

Example

$$\lim_{x\to 2} \frac{x^2+4x+5}{x^2+5}$$

$$\underset{(iv)}{=} \frac{\lim_{x\to 2}(x^2+4x+5)}{\lim_{x\to 2}(x^2+5)}$$

$$\underset{(i)}{=} \frac{\lim_{x\to 2}(x^2)+\lim_{x\to 2}(4x)+\lim_{x\to 2}(5)}{\lim_{x\to 2}(x^2)+\lim_{x\to 2}(5)}$$

$$\underset{(ii)\ \&\ (iii)}{=} \frac{\left(\lim_{x\to 2} x\right)^2+4\lim_{x\to 2} x+\lim_{x\to 2}(5)}{\left(\lim_{x\to 2} x\right)^2+\lim_{x\to 2}(5)}$$

$$= \frac{2^2+4\cdot 2+5}{2^2+5}$$

$$= \frac{17}{9}.$$

Could prove

$$\lim_{x\to 2} x = 2$$

$$\lim_{x\to 2}(5) = 5$$