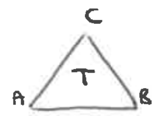
Lecture 1

0 Course info & Syllabus

1 Groups

1.1 Groups: the need for structure

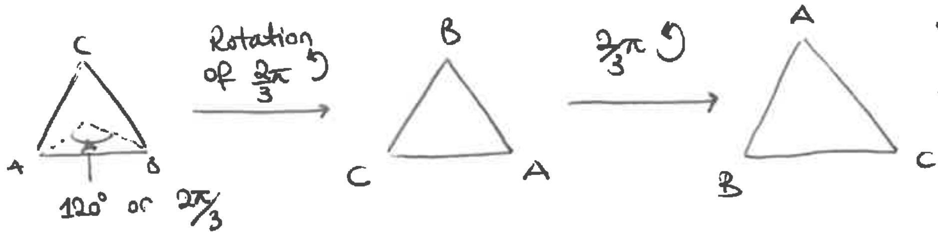
Consider an equilateral triangle T.



What operations can we perform on T in a way that the result of the operation leaves "unchanged" what we see?
In other words, what are the symmetries of T?

[Note: we give name to vertices only to mark which operation we are performing. Position in space does not matter]

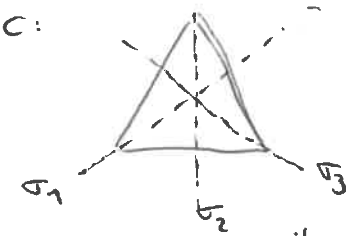
We can definitely rotate our triangle without changing what we see:



what happens if we apply one more the $2/3\pi$ rotation?

Observe that we can repeat this operation and again get the same figure.
We say that we can compose the two rotations (and thus get a rotation of $\frac{4}{3}\pi$).

What else can we perform? The triangle is symmetric w.r.t. the three axes through A, B, C:



This means, if we flip it along each of them we still see the same triangle. Call these operations $\sigma_1, \sigma_2, \sigma_3$.

Finally, we can leave it as it is, thus performing the identity operation.

In total the SET of symmetries of T is:

$$\{\text{identity}, \underbrace{\text{rotation of } 2/3\pi}_{=: \rho_{2/3\pi}}, \underbrace{\text{rotation of } 4/3\pi}_{=: \rho_{4/3\pi}}, \sigma_1, \sigma_2, \sigma_3\}$$ (we will call it $S(T)$)

So far we know:

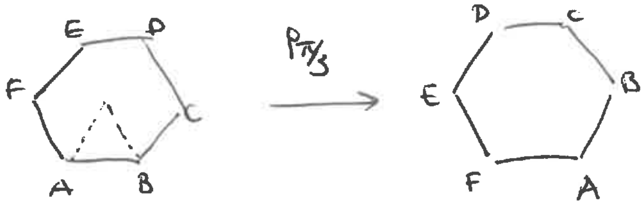
- $|S(T)| = 6$ (it contains 6 elements)
- Elements of $S(T)$ can be composed to get another element of $S(T)$

[check what you get by composing σ_1 and $\rho_{2/3\pi}$, or σ_2 with itself. Also: does the order of these compositions matter?]

Let's move on to another polygon and its symmetries.

Let's consider a regular hexagon H and the set of its rotations (so, no flipping allowed..) ②

Similarly to T, we note that a rotation of $2\pi/6$ $(= \pi/3)$ can be performed to H without changing "the shape that we see:



and so on re-applying $\rho_{\pi/3}$.

Again we note:

$$\rho_{\pi/3} \circ \rho_{\pi/3} = \rho_{2/3\pi}$$

$$\rho_{\pi/3} \circ \rho_{\pi/3} \circ \rho_{\pi/3} = \rho_{\pi} \quad \text{and so on.}$$

$$\underbrace{\rho_{\pi/3} \circ \ \circ \ \circ \ \circ \ \circ \rho_{\pi/3}}_{6 \text{ times}} = \rho_{2\pi} = \rho_0 = \text{identity operation}$$

we call $R(H)$ the SET of all rotations of H. We note

- $|R(H)| = 6$
- Elements in $R(H)$ can be composed to form another element of $R(H)$.

Question: Are $S(T)$ and $R(H)$ "the same"? As sets they are. But we know they encode symmetries and/or rotations of two different geometric objects. The GROUPS $(S(T), \circ)$ and $(R(H), \circ)$ are different as we will now see.

In this sense we can say that groups arise as algebraic structures describing symmetries

$S(T)$

$\parallel$ Here $\rho = \rho_{2/3\pi}$

$\{id, \rho, \rho \circ \rho, \sigma_1, \sigma_2, \sigma_3\}$

Every element can be repeatedly applied until it gives the identity

How often do we need to apply:

$\rho \dashrightarrow 3$

$\rho \circ \rho \dashrightarrow 3$

σ_1, σ_2 or $\sigma_3 \dashrightarrow 2$

to get the identity?

$R(H)$

$\parallel$ here $x = \rho_{\pi/3}$

$\{id, x, x \circ x, x \circ x \circ x, x \circ x \circ x \circ x, x \circ x \circ x \circ x \circ x\}$

same here.

How often do we need to apply

$x \dashrightarrow 6$

$x \circ x \dashrightarrow 3$

$x \circ x \circ x \dashrightarrow 2$

$x \circ x \circ x \circ x \dashrightarrow 3$

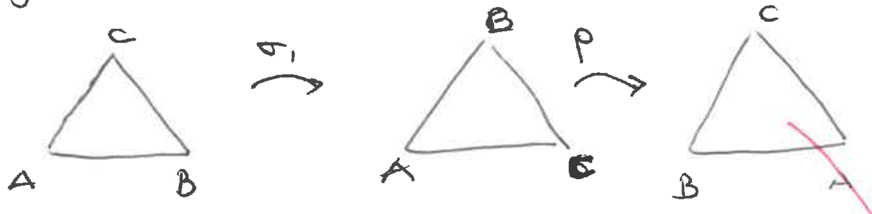
$x \circ x \circ x \circ x \circ x \dashrightarrow 6$

to get the identity?

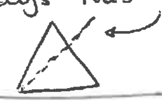
Here we see the first substantial diffrence:
In S(T) every element (apart from the identity) needs to be applied 2 or 3 times in order to get the identity. In R(H) we have 2 element which need to be applied 6 times in order to get the identity.

Also, if we try to compose any 2 elements of R(H) we will always find that the order in which we perform them doesn't matter.

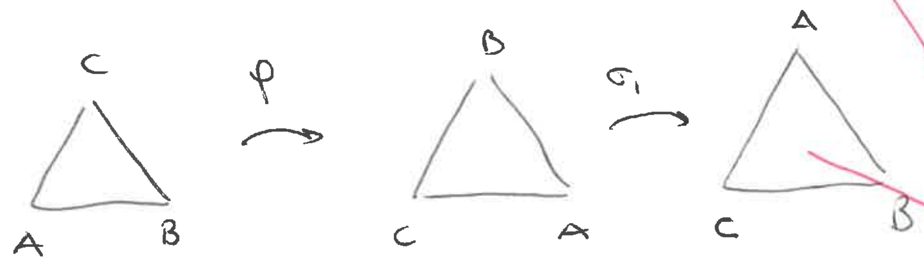
Let's go back to T and consider $\sigma_1 \circ \rho$ and $\rho \circ \sigma_1$ (*)



(*) we will think of the reflection axis as fixed on the plane on which the triangle lies: it is always this line



But



Different outcome!

We conclude that in S(T) the order in which we apply the elements matters.
We will say that (S(T), ∘) is a non-abelian or non-commutative group.

Now we should be all convinced that this structure "sees more" or "encodes more" than sets. Next: * define & study groups
* do we already know any groups?