Q: Does every vector space have a basis?

The way we defined them, bases are always finite. It turns out that some vector spaces are so "large", they don't have finite bases. Examples include $\mathbb{P}$ (polynomials of arbitrary degree), $\mathbb{S}$ (discrete signals), and $\mathcal{C}(\mathbb{R})$ (real-valued continuous functions on $\mathbb{R}$).

Rather than extend our concept of a basis to include infinite ones, we'll study vector spaces with finite bases in detail.

Defn A vector space V is finitely generated (or finite-dimensional) if $V = \operatorname{span}\{v_1, \dots, v_p\}$ for some $p \geq 0$ and $v_1, \dots, v_p \in V$.

(Here, for $p=0$, $\operatorname{span}\{\} := \{0\}$.)

"Casting out" lemma: Suppose that $V = \operatorname{span}\{v_1, \dots, v_p\}$ and that some v_k is a linear combination of the other vectors

$$v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_p.$$

Then $\boxed{V = \operatorname{span}\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_p\}.}$

We "cast out" v_k from our spanning set.

Thm: Let V be a finitely generated vector space with $V \neq \{0\}$.

Then V has a basis.

Indeed, here is a method for finding a basis of V:

- Let $V = \operatorname{span}\{v_1, \dots, v_p\}$. (Possible because V is finitely generated.)

- If no v_k belongs to $\operatorname{span}\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_p\}$, then $v_1, \dots, v_p$ are linearly independent and hence

$$(v_1, \dots, v_p)$$

is a basis of V.

- Otherwise, for some k, we have $v_k \in \operatorname{span}\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_p\}$.

We then discard v_k from our spanning set (this lowers p!) and repeat the steps above for the resulting smaller set.

- After finitely many iterations, we will have found a basis of V.

<u>Ex:</u> let $p_1(t) = 2t - t^2$, $p_2(t) = 2 + 2t$, $p_3(t) = 6 + 16t - 5t^2$

and let $V = \text{span}\{p_1(t), p_2(t), p_3(t)\} \subseteq \mathbb{P}_2$.

Goal: find a basis of V.

Since $\boxed{p_3(t) = 3p_2(t) + 5p_1(t)}$, $p_3(t) \in \text{span}\{p_1(t), p_2(t)\}$

so $V = \text{span}\{p_1(t), p_2(t)\}$. Next, neither one of $p_1(t)$ and $p_2(t)$ is a scalar multiple of the other. Hence, $p_1(t), p_2(t)$ are linearly independent and $(p_1(t), p_2(t))$ is a basis of V.

Bases of Nul A and Col A

Let A be an $m \times n$ matrix. Using row reduction, beginning with A, we obtain a (unique!) matrix A' in reduced row echelon form.

Recall:
- $\text{Nul}\,A = \text{Nul}\,A'$.
- We can read off a spanning set of $\text{Nul}\,A$ from A' (see lecture #5).

<u>Thm:</u> This method always produces a basis of $\text{Nul}\,A$.

Ex: Spse that $A \rightsquigarrow A' = \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ via row reduction.

Then $x = \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} \in \text{Nul}\, A' \iff \begin{array}{l} x_1 = 2x_2 + x_4 \\ \quad \& \\ x_3 = -2x_4 \end{array}$

$$\iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Hence, $\text{Nul}\, A = \text{Nul}\, A' = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ and $\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right)$

is a basis of this space.

Q: What about Col A? We know a spanning set (namely, the set of column vectors). Is there a better way to find a basis than to use the "casting out method"?

Fact: Let A be a matrix with associated matrix A' in reduced row echelon form. Then the columns of A and those of A' satisfy the "same linear dependence relations".

(Formally: $Ax = 0 \iff A'x = 0$.) In particular, the i^{th} column of A is a linear combination of some other columns iff the same is true for A'.