Def^n Let V be a vector space and let $v_1, \ldots, v_p \in V$.

We say that $v_1, \ldots, v_p$ are linearly independent if the equation

$$c_1 v_1 + \ldots + c_p v_p$$

for $c_1, \ldots, c_p \in \mathbb{R}$ only has the "trivial solution"

$$c_1 = \ldots = c_p = 0.$$

"linearly dependent" := not linearly independent

Ex:

- A single vector $v \in V$ is linearly independent iff $v \neq 0$.
  (Indeed, 0 is linearly dependent because $1 \cdot 0 = 0$.
  Conversely, let $v \in V$ be linearly dependent. Then there exists $c \in \mathbb{R}$, $c \neq 0$ with $cv = 0$. Hence,
  $$v = c^{-1}(cv) = c^{-1} 0 = 0.)$$
- No collection of vectors containing the zero vector is linearly independent.
- Let $u, v \in V$ with $u \neq 0 \neq v$. Then u, v are linearly independent iff $u \neq cv$ for all $c \in \mathbb{R}$.

Thm: Let $v_1, \dots, v_p \in V$. Then $v_1, \dots, v_p$ are linearly dependent iff $\exists j$ s.t. v_j is a linear combination of $v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_p$

Ex.: Recall that $\mathbb{P}_n$ = vector space of all polynomials $p(t)$ of degree $\leq n$

and $\mathbb{P} = \bigcup_{n=0}^{\infty} \mathbb{P}_n$ = vector space of all polynomials $p(t)$ (no degree constraints).

Let $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = 4-t$. Are $p_1(t), \dots, p_3(t)$ linearly independent in $\mathbb{P}$? No: $p_3(t) = 4p_1(t) + (-1)p_2(t)$ $\in \operatorname{span}\{p_1(t), p_2(t)\}$. Now use the Thm!

Problem (from 2018/19 exam paper) Decide (with justification) if t is a linear combination of the polynomials t^2-t and $2t+1$ in $\mathbb{P}_2$.

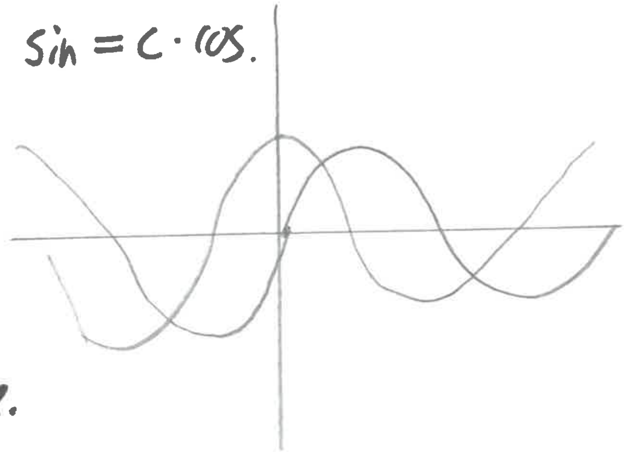
Soln: Suppose that t is a linear combination of t^2-t and $2t+1$.

Then $\exists c, d \in \mathbb{R}$ s.t.

$$0t^2+1t+0 = t = c(t^2-t) + d(2t+1)$$
$$= ct^2 + (2d-c)t + d$$

Polynomials are equal iff they have the same coefficients.
Hence, $c = d = 0$ and we obtain contradiction $t = 0$.
We conclude that t is not a linear combination of $t^2 - t$ and $2t + 1$.

Ex.: • sin, cos are linearly independent in $\underbrace{\mathcal{C}([0,1])}_{\substack{= \text{space of cont. fns} \\ [0,1] \to \mathbb{R}}}$
because neither function is a scalar multiple of the other. Indeed, suppose that $\sin = c \cdot \cos$.
Then $0 = \sin(0) = c \cdot \cos(0)$
$= c \cdot 1 = c$
so $\sin = 0$ which is nonsense.



• $\sin t \cdot \cos t$, $\sin(2t)$ are linearly dependent as vectors in $\mathcal{C}([0,1])$ because $\boxed{\sin(2t) = 2 \sin t \cdot \cos t.}$

• $\sin t \cdot \cos t$, $\cos(2t)$ are linearly independent in $\mathcal{C}([0,1])$.
We can use the argument from above: suppose that
$\sin t \cdot \cos t = c \cdot \cos(2t)$ for all $t \in [0,1]$. For $t = 0$,
we get $0 = \sin 0 \cdot \cos 0 = c \cos(0) = c$ and
hence the contradiction $\sin t \cdot \cos t = 0$ for all $t \in [0,1]$.

Defn: We say that a sequence of vectors $(v_1, \dots, v_p)$ in some vector space V is a basis of V if

① $v_1, \dots, v_p$ are linearly independent and ← "no redundancy"

② $V = \operatorname{span}\{v_1, \dots, v_p\}$. ← "get everything"

Ex:

- $\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right)$ is a basis of $\mathbb{R}^n$, called the standard basis.
- $(1, t, t^2, \dots, t^n)$ is a basis of $\mathbb{P}_n$. (Why?)

Remark:

- Many people define bases to be sets rather than sequences. For them, a set $\mathcal{B}$ is a basis of V iff $\mathcal{B} = \{v_1, \dots, v_p\}$ for linearly independent $v_1, \dots, v_p$ with $V = \operatorname{span}\{v_1, \dots, v_p\}$. The only real difference is that we keep track of the order of vectors in a basis.
- One can also consider spanning sets, linear independence, and bases for infinite collections of vectors. This is fun but a bit tricky!