Calculus provides natural & interesting examples of vector spaces and linear transformations.

Ex. (differentiation): let $a, b \in \mathbb{R}$ with $a < b$.

- Recall: $\mathcal{C}([a,b])$ is the vector space of continuous functions from $[a,b]$ to $\mathbb{R}$ with addition $(f+g)(x) = f(x)+g(x)$ and scalar multiplication $(cf)(x) = c\,f(x)$.
- let $\mathcal{C}'([a,b]) := \{ f \in \mathcal{C}([a,b]) : f' \text{ exists and is continuous on } [a,b] \}$.

Known results from calculus tell us that $\mathcal{C}'([a,b])$ is a subspace of $\mathcal{C}([a,b])$

- Moreover, the function $T : \mathcal{C}'([a,b]) \longrightarrow \mathcal{C}([a,b])$, $T(f) = f'$ is a linear transformation. (This is because $(f+g)' = f' + g'$ and $(cf)' = c f'$!)
- What is $\operatorname{Ker} T$? By definition, $f \in \operatorname{Ker} T$ iff $f' = 0$.

Calculus: $f' = 0$ iff f is constant so

$$\operatorname{Ker} T = \text{constant functions on } [a,b].$$

- What is Ran T? Claim: $\boxed{\operatorname{Ran} T = \mathcal{C}([a,b])}$

This is the Fundamental Thm of Calculus:

Every continuous function $f:[a,b]\to\mathbb{R}$ (i.e. $f\in\mathcal{C}([a,b])$) has an antiderivative, e.g. $F(x)=\int_a^x f(t)\,dt$ satisfies

$T(F)=f$.

Ex (integration): The function $\mathcal{C}([a,b])\longrightarrow\mathbb{R}$, $f\longmapsto\int_a^b f(x)\,dx$ is a linear transformation by the rules $\int_a^b (f(x)+g(x))\,dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx$,

etc.

Ex (sampling): Recall that $\mathbb{S}$ is the vector space of doubly infinite sequences $(y_k)=(\dots,y_{-1},y_0,y_1,\dots)$, so-called discrete signals.

In signal processing, sampling refers to reducing continuous signals to discrete ones. The sampling function $\mathcal{C}(\mathbb{R})\longrightarrow\mathbb{S}$,

$$f\longmapsto(f(k))=(\dots,f(-1),f(0),f(1),\dots)$$

is a linear transformation. Filters are functions $\mathbb{S}\to\mathbb{S}$ which e.g. block or amplify certain frequencies. A linear filter is a filter which is a linear transformation. Ex: a "volume knob" $(y_k)\mapsto(\lambda y_k)$ for fixed $\lambda>0$.

Problem (from 2018/19 exam paper):

Decide (with justification) if

$$b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

belongs to the column space of $A = \begin{bmatrix} 1 & 0 & -2 & -1 \\ -1 & 3 & 5 & 4 \\ 2 & 1 & -3 & -1 \end{bmatrix}$.

Soln: Row reduction

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 1 \\ -1 & 3 & 5 & 4 & 2 \\ 2 & 1 & -3 & -1 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 1 \\ 0 & 3 & 3 & 3 & 3 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right].$$

This is inconsistent so $b \notin \text{Col} A$.

Recall: abstract vector spaces don't come with "intrinsic" coordinates

Goal: (1) introduce some system of coordinates and

(2) understand how arbitrary these are.