Objective: Given an $m \times n$ matrix A and $b \in \mathbb{R}^m$, decide if $b \in \operatorname{Col} A$.

Ex: $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$. Is $b \in \operatorname{Col} A$?

Soln: We need to decide if $Ax = b$ has a soln $x \in \mathbb{R}^4$.

We use row reduction:

$$[A \mid b] = \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ 0 & -1 & 5 & 4 & 2 \\ 0 & 1 & -5 & 9/2 & -3/2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17/2 & 1/2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 1 & 1/17 \end{array}\right]$$

consistent!

so $Ax = b$ for $\boxed{x_4 = 1/17}$,

x_3 free, e.g. $\boxed{x_3 = 0}$

$\boxed{x_2 = -2 + 5x_3 + 4x_4 = -30/17}$

$2x_1 = 3 - 4x_2 + 2x_3 - x_4 = 10$

so $\boxed{x_1 = 5}$

Hence $\boxed{b = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} = A \begin{bmatrix} 5 \\ -30/17 \\ 0 \\ 1/17 \end{bmatrix} \in \operatorname{Col} A.}$

Q: Do these matrix computations (row reduction) and concepts (null space, column space) have analogues for general vector spaces? What is an "abstract matrix"?

Def^n Let V and W be vector spaces. A linear transformation T from V to W is a function $T: V \longrightarrow W$ (i.e. a "rule" which assigns a unique $T(x) \in W$ to each $x \in V$) such that

- $T(u+v) = T(u) + T(v) \quad \forall u, v \in V$ and
- $T(cu) = c\,T(u) \quad \forall u \in V,\ c \in \mathbb{R}$.

That is, a linear transformation is a function which "respects" or "is compatible with" the vector space structure.

Def^n Let $T: V \to W$ be a linear transformation.

- The kernel of T is $\boxed{\operatorname{Ker} T = \{u \in V : T(u) = 0\}.}$
- The range (or image) of T is $\boxed{\operatorname{Ran} T = \{T(u) : u \in V\}.}$

Ex: Let A be an $m\times n$ matrix. Define $T:\mathbb{R}^n\to\mathbb{R}^m$ via

$$T(x)=Ax \qquad (x\in\mathbb{R}^n).$$

① Then T is a linear transformation:

- $T(x+y)=A(x+y)=Ax+Ay = T(x)+T(y).$
- $T(cx)=A(cx)=c(Ax) = cT(x)$

(Here, we used familiar properties of matrix multiplication)

② $\operatorname{Ker} T=\{x\in\mathbb{R}^n : T(x)=Ax=0\} = \operatorname{Nul} A$

③ $\operatorname{Ran} T=\{T(x)=Ax : x\in\mathbb{R}^n\} = \operatorname{Col} A.$

In particular, $\operatorname{Ker} T$ is a subspace of $\mathbb{R}^n$ and $\operatorname{Ran} T$ is a subspace of $\mathbb{R}^m$.

Thm Let $T:V\to W$ be a linear transformation.

Then:
- $\operatorname{Ker} T$ is a subspace of V.
- $\operatorname{Ran} T$ is a subspace of W

For the curious: each of these constructions exhausts all subspaces.

Thm Let V be a vector space and let $H\subseteq V$ be a subspace. There are vector spaces U and W and linear transformations $S:U\to V$ and $T:V\to W$ with $\operatorname{Ran} S=H=\operatorname{Ker} T$. (This requires some ideas.)

We saw that every $m\times n$ matrix A gives rise to a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ via $T(x) = Ax$.

Q: Are there any other linear transformations $\mathbb{R}^n \to \mathbb{R}^m$?

Answer: No — linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ and $m\times n$ matrices are essentially the "same thing". Let's recover a matrix from a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$.

Let $e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th}\text{ row}$. Then each $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ is a linear combination of $e_1, \dots, e_n$:

$$x = \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{bmatrix}$$

$$= x_1 e_1 + \dots + x_n e_n.$$

Let $A := [\underbrace{T(e_1)}\ \cdots\ \underbrace{T(e_n)}]$, an $m\times n$ matrix.
in $\mathbb{R}^m$

Then $T(x) = T(x_1 e_1 + \dots + x_n e_n) \overset{T\text{ linear transf.}}{=} x_1 T(e_1) + \dots + x_n T(e_n)$

$$= Ax.$$

That is, T was obtained from the matrix A as above.