Two main sources of subspaces

- Spans of vectors ("from the bottom")
- "Kernels" / "null spaces" of "linear transformations"

↑ These things generalize sets of solutions to homogeneous systems of linear equations.

Defn Let A be an $m \times n$ matrix. The null space of A is

$$\boxed{\operatorname{Nul} A = \{x \in \mathbb{R}^n : Ax = 0\}}.$$

in $\mathbb{R}^m$!

Recall: for $A = [a_1 \cdots a_n]$ with $a_1, \ldots, a_n \in \mathbb{R}^m$, we have $Ax = x_1 a_1 + \cdots + x_n a_n$.

Ex: Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$.

Then $$A \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{so} \quad \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \in \operatorname{Nul} A.$$

On the other hand, $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \notin \operatorname{Nul} A$

Thm: $\mathrm{Nul}\,A$ is a subspace of $\mathbb{R}^n$

This follows easily from familiar properties, e.g. $A\underset{\substack{\uparrow\\ \text{in } \mathbb{R}^n}}{0} = \underset{\substack{\uparrow\\ \text{in } \mathbb{R}^m}}{0}$,

$A(x+y) = Ax + Ay$, and $A(cx) = c(Ax)$ for $x, y \in \mathbb{R}^n$

and $c \in \mathbb{R}$.

Curious contrast:
- Given A, it is very easy to test if a given vector belongs to $\mathrm{Nul}\,A$. (Just multiply!)
- But how can we find non-zero vectors in $\mathrm{Nul}\,A$ or prove that none exist?

Def^n A spanning set of a subspace H of V is a collection of vectors in V whose span is H.

Goal: Given A, find a finite spanning set of $\mathrm{Nul}\,A$.

Fact: Let A' be obtained from A by performing an elementary row operation:
- Multiply some row by a non-zero constant.
- Add a multiple of some row to another one.
- Interchange two rows.

Then $\mathrm{Nul}\,A = \mathrm{Nul}\,A'$.

Recall: A matrix is in reduced row echelon form if

- all non-zero rows are above all zero rows,
- the leading entry in each non-zero row is to the right of the leading entries of all rows above it, and
- each leading entry is 1 and is the only non-zero entry in its column.

Fact: Using elementary row operations, every matrix can be row reduced to obtain a unique (!) matrix in reduced row echelon form.

It turns out that we can read off a spanning set of $\mathrm{Nul}\,A$ from the reduced row echelon form of A.

Ex:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \underbrace{\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{reduced row echelon form}} =: A'.$$

We know: $\operatorname{Nul} A = \operatorname{Nul} A'$.

When does a vector belong to $\operatorname{Nul} A'$?

Clearly, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} \in \operatorname{Nul} A' \iff A'x = 0$

$$\iff x_1 - 2x_2 - x_4 + 3x_5 = 0 \ \& \ x_3 + 2x_4 - 2x_5 = 0$$

$$\iff \boxed{x_1 = 2x_2 + x_4 - 3x_5 \ \& \ x_3 = -2x_4 + 2x_5}$$

free variables x_2, x_4, x_5

$$\iff \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

So
$$\boxed{\mathrm{Nul}\,A = \operatorname{span}\left\{ \begin{bmatrix} 2\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 1\\0\\-2\\1\\0 \end{bmatrix}, \begin{bmatrix} -3\\0\\2\\0\\1 \end{bmatrix} \right\}.}$$

That is, we found a finite spanning set of $\mathrm{Nul}\,A$!

Apart from null spaces, matrices give rise to subspaces in another way.

Def^n Let $A = [a_1 \cdots a_n]$ be an $m\times n$ matrix (so $a_1, \dots, a_n \in \mathbb{R}^m$).

The column space of A is $\boxed{\mathrm{Col}\,A := \operatorname{span}\{a_1, \dots, a_n\}.}$

Note that $\mathrm{Nul}\,A$ is a subspace of $\mathbb{R}^n$ while $\mathrm{Col}\,A$ is a subspace of $\mathbb{R}^m$.

Exercise: $\mathrm{Col}\,A = \{Ax : x \in \mathbb{R}^n\} = \{b \in \mathbb{R}^m : \exists x \in \mathbb{R}^n.\ b = Ax\}$.

Given A, we thus obtain two vector spaces:

$\mathrm{Nul}\,A$
- easy to test membership: does $x \in \mathbb{R}^n$ belong to $\mathrm{Nul}\,A$?
- need to work to produce a (finite) spanning set

$\mathrm{Col}\,A$
- defined via spanning set
- need to work to test membership

Q: how can we do this?