<u>Recall:</u> A subspace of a vector space is a subset which forms a vector space in its own right w.r.t. the operations inherited from the ambient vector space.

<u>Q:</u> How do subspaces arise in nature?

- "From the top": all vectors with "suitable properties" (e.g. solutions to suitable equations)
- "From the bottom": start with a collection of vectors and consider the subspace they "<u>span</u>" ⇝ soon!

<u>Ex:</u> Let $D \subseteq \mathbb{R}$ be a subset. Let V be the vector space of all functions $f: D \to \mathbb{R}$ from before.

(Reminder: $(f+g)(x) = f(x) + g(x)$, $(cf)(x) = c f(x)$; $(-f)(x) = -f(x)$, the zero vector is the function $0(x) = 0$.)

Let $\mathcal{C}(D) := \{ f: D \to \mathbb{R} : \underline{f \text{ is continuous}} \}$.

$\Leftrightarrow f$ preserves limits

<u>Thm:</u> $\mathcal{C}(D)$ is a subspace of V.

Why? Calculus!

- Constant functions are continuous. In particular, the zero vector of V (= the constant function with value zero) belongs to $\mathcal{C}(D)$.
- Sums and products of continuous functions are continuous.

Hence, $\mathcal{C}(D)$ is closed under addition and scalar multiplication.

Subspaces spanned by vectors

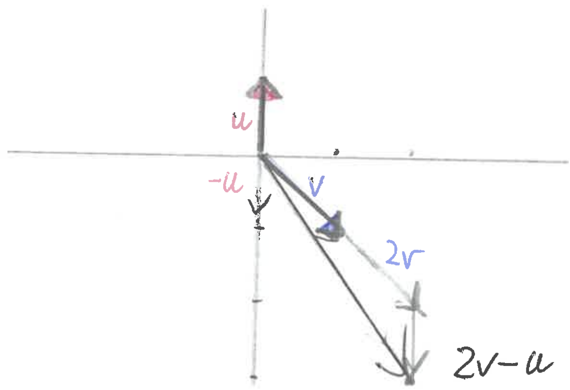
Def^n A *linear combination* of vectors $u_1, \dots, u_p$ in some vector space is a vector of the form

$$c_1 u_1 + \dots + c_p u_p$$

for scalars $c_1, \dots, c_p \in \mathbb{R}$.

Ex: • In $\mathbb{R}^2$, $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Indeed, $\begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



- Is $\begin{bmatrix}1\\1\end{bmatrix}$ a linear combination of $\begin{bmatrix}2\\3\end{bmatrix}$ and $\begin{bmatrix}-4\\-6\end{bmatrix}$ in $\mathbb{R}^2$?

Let's suppose that it is. Then there are $c, d \in \mathbb{R}$ with

$$\begin{bmatrix}1\\1\end{bmatrix} \overset{!}{=} c\begin{bmatrix}2\\3\end{bmatrix} + d\begin{bmatrix}-4\\-6\end{bmatrix}$$

$$= \begin{bmatrix}2c-4d\\3c-6d\end{bmatrix} = \begin{bmatrix}2(c-2d)\\3(c-2d)\end{bmatrix}.$$

In other words:

$$2(c-2d) = 1 = 3(c-2d)$$

so $c-2d = \frac{1}{2}$ but also $c-2d = \frac{1}{3}$ which is absurd. This contradiction shows that $\begin{bmatrix}1\\1\end{bmatrix}$ is not a linear combination of $\begin{bmatrix}2\\3\end{bmatrix}$ and $\begin{bmatrix}-4\\-6\end{bmatrix}$.

Defn Given vectors $u_1, \dots, u_p$ in some vector space V, their span is

$$\text{span}\{u_1, \dots, u_p\} := \{c_1u_1 + \dots + c_pu_p : c_1, \dots, c_p \in \mathbb{R}\}$$

$=$ set of all linear combinations of $u_1, \dots, u_p$.

Thm span$\{u_1, \dots, u_p\}$ is a subspace of V.

Note: In fact, span$\{u_1, \ldots, u_p\}$ is the "smallest" subspace of V that contains the vectors $u_1, \ldots, u_p$.

Immediate consequences:

- Every choice of vectors $u_1, \ldots, u_p$ provides us with an example of a subspace of V.
  (However, different sequences of vectors can span the same subspace!)
- If we recognize a subset of V as a span, then we know it's a subspace.

Ex: Let

$$H = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Claim: H is a subspace of $\mathbb{R}^4$.

Indeed,

$$H = \left\{ a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a subspace by the Thm.

Problem (from 2018/19 exam paper):

Find vectors $u, v, w \in V$ with $V = \text{span}\{u, v, w\}$, where V is the subspace of $\mathbb{R}^4$ consisting of all vectors of the form

$$\begin{bmatrix} 2a - c \\ -a \\ b + c \\ a - b \end{bmatrix}$$

for $a, b, c \in \mathbb{R}$.

Sol^n:
$$V = \left\{ \begin{bmatrix} 2a - c \\ -a \\ b + c \\ a - b \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$= \left\{ a \underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{u} + b \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}}_{v} + c \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{w} : a, b, c \in \mathbb{R} \right\}$$

$$= \text{span}\{u, v, w\}.$$

Non-ex: Let $H = \left\{ \begin{bmatrix} 3s \\ 2 + 5s \end{bmatrix} : s \in \mathbb{R} \right\}$. Is this a subspace of $\mathbb{R}^2$?

No! For example, $0 \notin H$. Suppose that $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3s \\ 2 + 5s \end{bmatrix}$ for $s \in \mathbb{R}$. Then $0 = 3s$ so $s = 0$ but then $2 + 5s = 2 \neq 0$, a contradiction.

Q: Is every subspace the span of some vectors?