Recall: A vector space is a collection of things that can be added together and multiplied by scalars subject to certain rules.

Today: We'll look at vector spaces sitting inside other vector spaces, so-called subspaces.

Def^n Let V be a vector space. A subspace of V is a subset of V which forms a vector space w.r.t. the addition and scalar multiplication inherited from V.

Fact: Let $H \subseteq V$. Then H is a subspace of V if and only if the following conditions are all satisfied:

- $0 \in H$
  the zero vector in V
- H is closed under addition in V, i.e. for all $u, v \in H$, we have $u + v \in H$.
- H is closed under multiplication by scalars, i.e. for all $u \in H$ and $c \in \mathbb{R}$, we have $cu \in H$.

Ex: • $\{0\}$ is a subspace of V for any vector space V.

• V —— —— —— ——

(These are boring subspaces, in general.)

Ex: Let $H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathbb{R} \text{ with } x+y+z=0 \right\}$.

Then H is a subspace of $\mathbb{R}^3$:

• First, clearly $H \subseteq \mathbb{R}^3$.

• Next, $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H$ because $0+0+0=0$.

• Let $u, v \in H$, say $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $v = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$.

We need to show that $u + v = \begin{bmatrix} x+x' \\ y+y' \\ z+z' \end{bmatrix} \in H$ which

is equivalent to

$$(x+x') + (y+y') + (z+z') = 0.$$

Since $u \in H$, we have $x + y + z = 0$.

—— $v \in H$, —— $x' + y' + z' = 0$.

Hence, $(x+y+z) + (x'+y'+z') = (x+x') + (y+y') + (z+z') = 0 + 0$
$= 0$

and thus $u+v \in H$.

- let $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in H$ and $c \in \mathbb{R}$.

Then: $x+y+z=0 \overset{\uparrow}{\Longrightarrow} c(x+y+z)=0$

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$$\Longrightarrow cx + cy + cz = 0$$

$$\rightsquigarrow \quad cu = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \in H.$$

Problem (from 2018/19 exam paper):

Decide (with justification) whether

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x+y \geq 0 \right\}$$

is a subspace of $\mathbb{R}^2$.

Soln: No, because H is not closed under multiplication by scalars. For instance, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in H$ (as $1+0 \geq 0$) but

$$(-1) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \notin H \text{ (as } -1+0 < 0).$$

(Note, however, that $\begin{bmatrix}0\\0\end{bmatrix} \in H$ and that H is closed under addition.)

Ex: $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ — in fact, it is not even a subset. However, $\mathbb{R}^2$ "looks like"

$$H := \left\{ \begin{bmatrix}x\\y\\0\end{bmatrix} : x, y \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$$

and H is a subspace. Indeed, $\begin{bmatrix}0\\0\\0\end{bmatrix} \in H$. Let

$u = \begin{bmatrix}x\\y\\0\end{bmatrix}, v = \begin{bmatrix}x'\\y'\\0\end{bmatrix} \in H$ and $c \in \mathbb{R}$. Then

$$u + v = \begin{bmatrix}x\\y\\0\end{bmatrix} + \begin{bmatrix}x'\\y'\\0\end{bmatrix} = \begin{bmatrix}x+x'\\y+y'\\0\end{bmatrix} \in H$$

and

$$cu = \begin{bmatrix}cx\\cy\\0\end{bmatrix} \in H.$$

Def^n: $\mathbb{P} := \bigcup_{n=0}^{\infty} \mathbb{P}_n = \left\{ p(t) = a_0 + a_1 t + \dots + a_n t^n : n \in \mathbb{N}_0, a_0, \dots, a_n \in \mathbb{R} \right\}$

is the vector space of all polynomials; addition and scalar multiplication are defined as in $\mathbb{P}_n$.

<u>Fact:</u> • $\mathbb{P}_m$ is a subspace of $\mathbb{P}_n$ for $m \leq n$.

• $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ for each $n \geq 0$

Note that $\mathbb{P}_0 = \mathbb{R} =$ constant polynomials

$\stackrel{!}{=} \{ p(t) \in \mathbb{P} : p'(t) = 0 \}$,

where, for $p(t) = a_0 + a_1 t + \dots + a_n t^n$, we have

$$p'(t) = a_1 + 2a_2 t + \dots + n a_n t^{n-1}$$

is the derivative of $p(t)$.

More generally,

$$\mathbb{P}_n = \{ p(t) \in \mathbb{P} : p^{(n+1)}(t) = 0 \}$$

so these subspaces of $\mathbb{P}$ can be described as solutions of certain equations. This will be a central theme!

<u>Q:</u> Can we describe all subspaces of a vector space?

<u>Exercise:</u> There are precisely three types of subspaces of $\mathbb{R}^2$: (1) $\{0\}$, (2) $\mathbb{R}^2$, and (3) lines through the origin.