NUI Galway
OÉ Gaillimh

Semester I Examinations 2018/2019

Exam Codes	3BA1, 4BA4, 4BCW1
Exams	Bachelor of Arts, Arts (International), and Creative Writing
Module	Linear Algebra I
Module Code	MA313
External Examiner	Prof. T. Brady
Internal Examiners	Prof. G. Ellis
	Dr T. Rossmann
Instructions	**Answer all four questions.**
	Each question carries 25 marks.
Duration	2 hours
No. of Pages	3 pages, including this one
School	Mathematics, Statistics and Applied Mathematics
Requirements:	No special requirements
Release to Library	Yes
MCQ	No
Statistical / Log Tables	No

Q1. (a) Define what is meant by a *subspace* of a vector space. Give an example of a vector space V and a subspace H of V with $H \neq V$. [5 marks]

(b) Decide (with justification) whether

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x + y \geqslant 0 \right\}$$

is a subspace of $\mathbb{R}^2$. [5 marks]

(c) Define the *span* of a sequence of vectors within a vector space. Find vectors $u, v, w \in V$ with $V = \text{span}\{u, v, w\}$, where V is the subspace of $\mathbb{R}^4$ consisting of all vectors of the form

$$\begin{bmatrix} 2a - c \\ -a \\ b + c \\ a - b \end{bmatrix}$$

for $a, b, c \in \mathbb{R}$. [5 marks]

(d) Decide (with justification) if

$$b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

belongs to the column space of

$$A = \begin{bmatrix} 1 & 0 & -2 & -1 \\ -1 & 3 & 5 & 4 \\ 2 & 1 & -3 & -1 \end{bmatrix}.$$

[10 marks]

Q2. Recall that $\mathbb{P}_n$ denotes the vector space of polynomials $p(t)$ of degree at most n.

(a) Define what is meant by a *linear combination* of a sequence of vectors. Decide (with justification) if t is a linear combination of the polynomials $t^2 - t$ and $2t + 1$ in $\mathbb{P}_2$. [5 marks]

(b) Define the terms (i) *linearly independent* sequence of vectors, (ii) *basis* of a vector space, (iii) *coordinate vector* of a vector relative to a basis, and (iv) *dimension* of a vector space. What is the dimension of $\mathbb{P}_n$? [5 marks]

(c) Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} p + 2q \\ -p \\ 3p - q \\ p + q \end{bmatrix} : p, q \in \mathbb{R} \right\}$$

of $\mathbb{R}^4$. [5 marks]

(d) Show that

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right)$$

is a basis of $\mathbb{R}^3$. Moreover, find the coordinate vector of

$$y = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

relative to $\mathcal{B}$. [10 marks]

Q3. (a) Define the terms (i) *linear transformation*, (ii) *kernel*, (iii) *range*, and (iv) *isomorphism*. Give an example of an isomorphism between two vector spaces. [5 marks]

(b) Define the terms (i) *null space*, (ii) *column space*, (iii) *rank*, and (iv) *nullity* of a matrix and (v) state the *Rank-Nullity Theorem*. [5 marks]

(c) i. What is the largest possible rank of a 4×7 matrix?

ii. What is the largest possible rank of a 7×4 matrix?

iii. If the null space of a 4×7 matrix A is 3-dimensional, what is the dimension of its column space?

[5 marks]

(d) Find bases for the null space and the column space of

$$A = \begin{bmatrix} 1 & -3 & -2 & 4 \\ 2 & 0 & 2 & 2 \\ 0 & 4 & 4 & -4 \end{bmatrix}.$$

[10 marks]

Q4. (a) Define the *length* of a vector in $\mathbb{R}^n$ and give an example of a vector in $\mathbb{R}^3$ of length 3. [5 marks]

(b) Define the term *orthogonal matrix* and give an example of an orthogonal 2×2 matrix with first row $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$. [5 marks]

(c) Find the orthogonal projection of $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ onto the line passing through $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and the origin in $\mathbb{R}^2$. [5 marks]

(d) Briefly indicate how vector spaces and linear transformations arise in signal processing. [5 marks]

(e) State the general *linear least-squares problem* and describe a method for solving it. [5 marks]