<u>Last time:</u> • Formal def^n of a "vector space".

• We showed that $\mathbb{R}^n$ is a vector space (with the usual definitions of vector addition & scalar multiplication)

<u>Today:</u> • Some examples of vector spaces that are quite unlike $\mathbb{R}^n$.

• Some common features of all vector spaces

<u>Ex. (discrete signals):</u>

Let $\mathbb{S}$ = doubly infinite sequences of real numbers

$$= \{ (y_k) = (\ldots, y_{-1}, y_0, y_1, \ldots) \ : \ y_k \in \mathbb{R} \ \forall k \in \mathbb{Z} \}.$$

Define

$$(y_k) + (z_k) := (y_k + z_k)$$

and

$$c\,(y_k) := (c\,y_k),$$

where $(y_k), (z_k) \in \mathbb{S}$ and $c \in \mathbb{R}$.

We call the elements of $\mathbb{S}$ $\boxed{\text{discrete signals.}}$

<u>Claim:</u> $\mathbb{S}$ (together with the operations defined above) is a vector space.

Prove it yourself!

For instance, the zero vector of $\mathbb{S}$ is

$$(0) = (\dots, 0, 0, \dots) = \text{all zero sequence}$$

because

$$(y_k) + (0) = (y_k + 0) = (y_k)$$

for each $(y_k) \in \mathbb{S}$

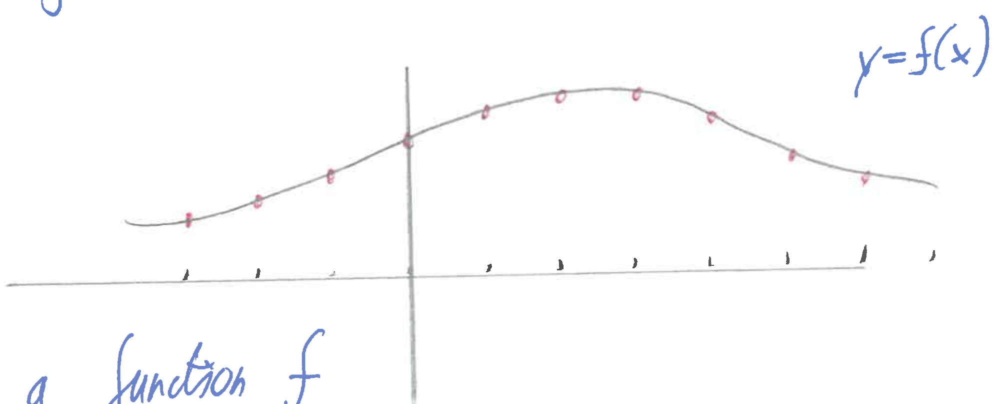
Also,

$$(y_k) + (-y_k) = (y_k + (-y_k)) = (0)$$

so $-(y_k) := (-y_k)$ satisfies axiom V4,

etc.

Why "discrete signals"? $\rightsquigarrow$ Measure continuous signals at discrete times.



More formally, a function f is sent to the discrete signal $(\dots, f(-1), f(0), f(1), \dots) = (f(k))$.

The vector space $\mathcal{S}$ is "unlike" any $\mathbb{R}^n$.

For instance, there is no finite collection of "coordinates" that can be used to distinguish between different elements in all cases.

(We'll understand this better later on!)

<u>Q:</u> What can we do with a general vector space?

Recall: addition & scalar multiplication are just some (unknown) rules subject only to the axioms (V1)–(V8).

How "exotic" can vector spaces be?

<u>Claim:</u> In any vector space V, for all $u \in V$ and $c \in \mathbb{R}$

- $\underbrace{0}_{\text{in } \mathbb{R}} u = \underbrace{0}_{\text{in } V}$
- $-u = (-1)u$

  (So negatives are unique!)
- $c0 = 0$

  zero vector of V
- The zero vector is unique.

Why?

- $0u = (0+0)u \underset{V6}{=} 0u + 0u$

$\rightsquigarrow$ add $-(0u)$

$$0 \underset{V4}{=} 0u + (-(0u))$$
$$= (0u + 0u) + (-(0u))$$
$$\underset{V2}{=} 0u + \underbrace{(0u + (-(0u)))}_{=0 \text{ by } V4}$$
$$= 0u + 0 \underset{V3}{=} 0u$$

- $c0 = c(0+0) = c0 + c0.$

Now add $-(c0)$ and proceed as before

etc.

Let's look at some further important examples of vector spaces.

<u>Ex. (polynomials):</u> For an integer $n \geq 0$, let $\mathbb{P}_n$ consist of all polynomials

$$p(t) = a_0 + a_1 t + \dots + a_n t^n$$

of degree $\leq n$, where $a_0, \dots, a_n \in \mathbb{R}$.

We can add polynomials in $\mathbb{P}_n$ in the usual way:

$$(a_0 + a_1 t + \dots + a_n t^n) + (b_0 + b_1 t + \dots + b_n t^n)$$
$$= (a_0 + b_0) + (a_1 + b_1) t + \dots + (a_n + b_n) t^n.$$

We further have

$$c\, p(t) = c a_0 + c a_1 t + \dots + c a_n t^n,$$

where $c \in \mathbb{R}$ and $p(t) = a_0 + a_1 t + \dots + a_n t^n$ as above.
These operations turn $\mathbb{P}_n$ into a vector space.
Why? Again, this boils down to properties of real numbers.

<u>Ex. (function spaces)</u>: Let $\mathbb{D}$ be a set (completely arbitrary!).
Let V be the set of all functions $f: \mathbb{D} \to \mathbb{R}$. Given $f, g \in V$ and $c \in \mathbb{R}$, we define $f+g \in V$ and $cf \in V$ via

$$(f+g)(x) := f(x) + g(x) \quad \text{and}$$
$$(cf)(x) := c f(x)$$

for $x \in \mathbb{D}$. These operations turn V into a vector space.
(Note: calculus lives in this space for $\mathbb{D} = \mathbb{R}, (a,b), \dots$)