Hence: $\hat{v} = \frac{v \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}.$$

Note that $v - \hat{v} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}$ is really orthogonal to u_1 & u_2.

Q: Does every subspace of $\mathbb{R}^n$ have an orthogonal basis?

(Note how this is similar to asking whether every vector space has a basis)

Thm Let $(v_1, \dots, v_p)$ be a basis of a subspace W of $\mathbb{R}^n$.

Define $u_1, \dots, u_p$ via

$$\boxed{u_1 := v_1}$$

$$\boxed{u_2 := v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1}$$

$$\boxed{u_3 := v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2}$$

$$\vdots$$

$$\boxed{u_p := v_p - \frac{v_p \cdot u_1}{u_1 \cdot u_1} u_1 - \dots - \frac{v_p \cdot u_{p-1}}{u_{p-1} \cdot u_{p-1}} u_{p-1}}$$

"Gram-Schmidt process"

Then $(u_1, \dots, u_p)$ is an orthogonal basis of W.

Ex let $W = \text{span}\{v_1, v_2\}$ for $v_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

We construct an orthogonal basis of W by using the Gram-Schmidt process. First, (v_1, v_2) is a basis of W. (Why?)

Next:
- $u_1 := v_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$
- $u_2 := v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{15}{45} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

(Note that indeed $u_1 \perp u_2$!)

Hence: $\left(\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right)$ is an orthogonal basis of W.

Orthonormal sets & orthogonal matrices

Recall: $(u_1, \dots, u_p)$ are orthogonal if $u_i \perp u_j$ for $i \neq j$.

We say that $u_1, \dots, u_p$ are orthonormal if they are orthogonal unit vectors (i.e. $\|u_i\| = 1$ for $i = 1, \dots, p$).

Easy: If $u_1, \dots, u_p$ are non-zero & orthogonal, then $\frac{1}{\|u_1\|} u_1, \dots, \frac{1}{\|u_p\|} u_p$ are orthonormal.

An orthonormal basis is a basis consisting of orthonormal vectors.

Ex: The standard basis of $\mathbb{R}^n$ is orthonormal.