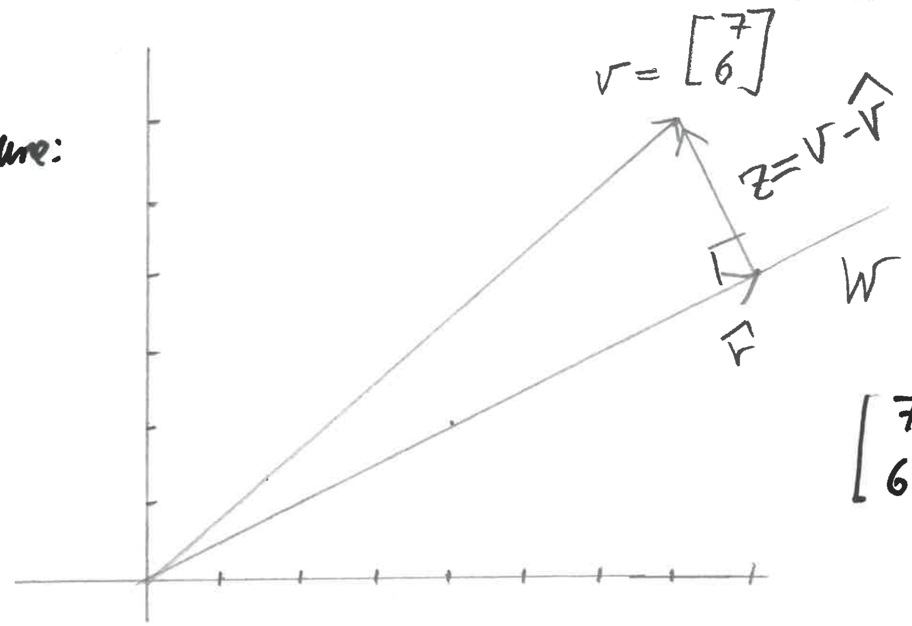
Ex let $v = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $W = \operatorname{span}\left\{ \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$ = the line through $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and the origin

Picture:



$$\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \underbrace{2 \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}}_{\in W} + \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\in W^\perp}$$

How can we compute $\hat{v}$ in general? We first consider a special case.

Lemma let $W = \operatorname{span}\{u\} \leq \mathbb{R}^n$ (for $0 \neq u \in \mathbb{R}^n$) be a line through the origin. Then the orthogonal projection $\hat{v}$ of $v \in \mathbb{R}^n$ on W

1)
$$\boxed{\hat{v} = \frac{v \cdot u}{u \cdot u} u.}$$

In our example above, $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ so $u \cdot u = 20$, $v \cdot u = 40$, and $\hat{v} = \frac{40}{20} u = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$.

Next: compute $\hat{v}$ for a general subspace W of $\mathbb{R}^n$ (not just lines)

Defn • A sequence of vectors $u_1, \dots, u_p \in \mathbb{R}^n$ is orthogonal if $u_i \perp u_j$ for all $i \neq j$.

• An orthogonal basis of a subspace of $\mathbb{R}^n$ is a basis which is orthogonal.

Fact If $u_1, \dots, u_p$ is an orthogonal sequence of non-zero vectors, then these vectors are linearly independent.

Thm Let $(u_1, \dots, u_p)$ be an orthogonal basis of a subspace W of $\mathbb{R}^n$. Then the orthogonal projection of $v \in \mathbb{R}^n$ onto W is given by

$$\boxed{\hat{v} = \frac{v \cdot u_1}{u_1 \cdot u_1} u_1 + \dots + \frac{v \cdot u_p}{u_p \cdot u_p} u_p.}$$

Remark If $W = \operatorname{span}\{u\}$ for $u \neq 0$, then (u) is an orthogonal basis of W and the Thm reduces to the identity $\hat{v} = \frac{v \cdot u}{u \cdot u} u$ from before.

Ex Let $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $W = \operatorname{span}\{u_1, u_2\}$

Since $u_1 \cdot u_2 = 2(-2) + 5 \cdot 1 + (-1) \cdot 1 = 0$ so $u_1 \perp u_2$ and (u_1, u_2) is an orthogonal basis of W.