Orthogonality

Recall: The inner product (or dot product) of vectors in $\mathbb{R}^n$ is given by

$$\boxed{\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + \dots + u_n v_n \in \mathbb{R}.}$$

Ex: $\begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} = 2 \cdot 3 + (-5) \cdot 2 + (-1) \cdot (-3) = -1$

Fact: For all $u, v, w \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

$$\left.\begin{array}{l} \bullet\ u \cdot v = v \cdot u \\ \bullet\ (u+v) \cdot w = u \cdot w + v \cdot w \\ \bullet\ (cu) \cdot v = u \cdot (cv) = c(u \cdot v) \end{array}\right\} \text{Easily verified by direct computation}$$

- $u \cdot u \geq 0$ and $u \cdot u = 0$ iff $u = 0$.

(Why? $u \cdot u = u_1^2 + \dots + u_n^2$.)

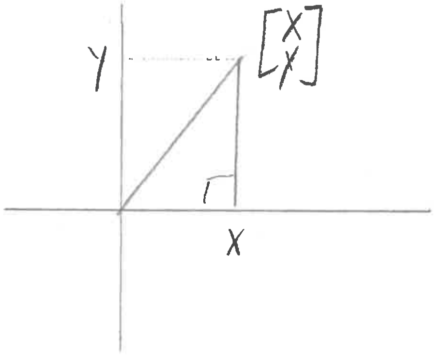
Defn The length (or norm) of a vector $v \in \mathbb{R}^n$ is

$$\boxed{\|v\| := \sqrt{v \cdot v} = \sqrt{v_1^2 + \dots + v_n^2} \geq 0.}$$

Fact $\|cv\| = |c| \cdot \|v\|$ for $c \in \mathbb{R}$ and $v \in \mathbb{R}^n$.

(Why? $\|cv\| = \sqrt{cv \cdot cv} = \sqrt{c^2 (v \cdot v)} = \sqrt{c^2}\,\sqrt{v \cdot v} = |c| \cdot \|v\|$.)

In $\mathbb{R}^2$:



$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \sqrt{x^2+y^2}$$

This agrees with Pythagoras

Defn The distance between $u, v \in \mathbb{R}^n$ is

$$\boxed{\operatorname{dist}(u,v) := \|u-v\|.}$$

Ex:

- $\left\| \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\| = \sqrt{3^2+2^2+1^2} = \sqrt{14}$
- $\operatorname{dist}\left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) = \left\| \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\|$

$$= \sqrt{2^2+0^2+2^2} = \sqrt{8} = 2\sqrt{2}.$$

Thm (Triangle inequality) $\|u+v\| \leq \|u\| + \|v\|$ for all $u, v \in \mathbb{R}^n$.

The distance function has the following key properties.

Let $u, v, w \in \mathbb{R}^n$. Then:

- $\operatorname{dist}(u,v) = 0 \iff u = v$.
- $\operatorname{dist}(u,v) = \operatorname{dist}(v,u)$.
- $\operatorname{dist}(u,w) \leq \operatorname{dist}(u,v) + \operatorname{dist}(v,w)$

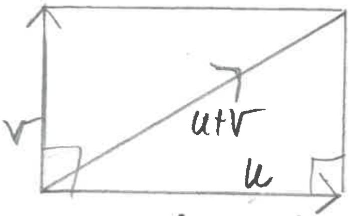
Defn We say that $u, v \in \mathbb{R}^n$ are orthogonal if $u \cdot v = 0$.

Notation: $u \perp v \iff u \cdot v = 0$.

Pythagorean Thm: If $u \perp v$, then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

Pf: $\|u+v\|^2 = (u+v)\cdot(u+v) = u\cdot u + \underbrace{u\cdot v}_{=0} + \underbrace{v\cdot u}_{=0} + v\cdot v$

$= \|u\|^2 + \|v\|^2$. //



Defn A unit vector is any vector v with $\boxed{\|v\| = 1}$.

Note that given a nonzero $v \in \mathbb{R}^n$, the vector $\frac{1}{\|v\|} v$ is a unit vector "in the same direction" as v.

(Why? $\left\| \frac{1}{\underbrace{\|v\|}_{\neq 0}} v \right\| = \left| \frac{1}{\|v\|} \right| \cdot \|v\| = \frac{\|v\|}{\|v\|} = 1$.)

Orthogonal projections and complements

Let W be a subspace of $\mathbb{R}^n$. We say that a vector $z \in \mathbb{R}^n$ is orthogonal to W if $z \perp w$ for all $w \in W$.

The orthogonal complement of W is

$$\boxed{W^\perp = \{ z \in \mathbb{R}^n : z \perp w \text{ for all } w \in W \}.}$$

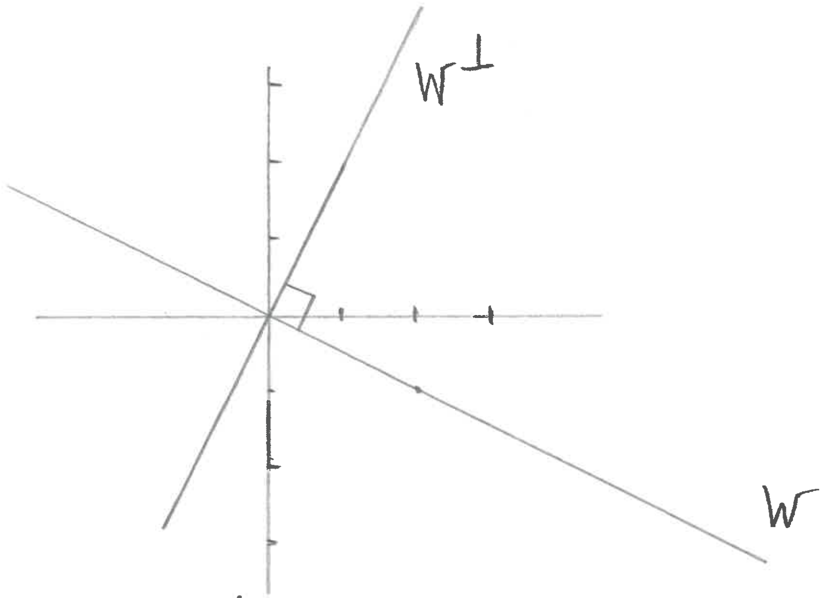
Ex: Let $W = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 2t \\ -t \end{bmatrix} : t \in \mathbb{R} \right\}$.

Then $\begin{bmatrix} x \\ y \end{bmatrix} \in W^\perp$ iff $\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 2t \\ -t \end{bmatrix} = 2xt - yt = (2x-y)t \overset{!}{=} 0 \quad \forall t$

iff $2x = y$.

Hence: $W^\perp = \left\{ \begin{bmatrix} x \\ 2x \end{bmatrix} : x \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, a subspace!

Picture:



Thm Let $W \subseteq \mathbb{R}^n$ be a subspace.

- $W^\perp$ is a subspace of $\mathbb{R}^n$.
- If $W = \operatorname{span}\{w_1, \dots, w_r\}$, then $W^\perp = \{ z \in \mathbb{R}^n : z \perp w_1, \dots, z \perp w_r \}$.
- Every vector $v \in \mathbb{R}^n$ has a unique representation

$$\boxed{v = \hat{v} + z}$$

for $\hat{v} \in W$ and $z \in W^\perp$.
- The function $\mathbb{R}^n \to W$, $v \mapsto \hat{v}$ is a linear transformation.
(It is called the orthogonal projection of $\mathbb{R}^n$ onto W.)
- $W \cap W^\perp = \{0\}$
- $\dim(W^\perp) = n - \dim(W)$.