Exam problems (from 2018/19):

- Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} p+2q \\ -p \\ 3p-q \\ p+q \end{bmatrix} : p, q \in \mathbb{R} \right\}$$

of $\mathbb{R}^4$.

Soln:
$$\begin{bmatrix} p+2q \\ -p \\ 3p-q \\ p+q \end{bmatrix} = p \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix} + q \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \text{ so } H = \operatorname{span} \left\{ \underset{u}{\begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}}, \underset{v}{\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}} \right\}.$$

Since $u \neq 0 \neq v$ and $u \neq cv$ for all $c \in \mathbb{R}$, (u, v) is a basis of H. Hence, $\dim H = 2$.

- Show that
$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right)$$

is a basis of $\mathbb{R}^3$. Moreover, find the coordinate vector of

$$y = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

relative to $\mathcal{B}$.

Soln: We do both tasks at once!

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 3 & 8 & 3 \end{bmatrix}}_{=:A} \left| \underbrace{\begin{matrix} 0 \\ 0 \\ -2 \end{matrix}}_{y} \right] \rightsquigarrow \left[\begin{matrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 0 & 2 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \\ -2 \end{matrix}\right]$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightsquigarrow \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \begin{matrix} 1 \\ -1 \\ 1 \end{matrix}\right]$$

Hence: • $\operatorname{rank} A = 3 \implies$ the columns of A form a basis of $\mathbb{R}^3$

$\implies \mathcal{B}$ is a basis of $\mathbb{R}^3$

Remember: The columns of an $n \times n$ matrix A form a basis of $\mathbb{R}^n \iff \operatorname{rank} A = n$.

(Not true for non-square matrices!)

• $A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$ so $[y]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Let's check this:

$$1 \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} = y.$$

An application of the theory that we developed:

What are the possible ranks of an $m\times n$ matrix A?

- Because $\operatorname{Col} A$ is a subspace of $\mathbb{R}^m$,

$$\operatorname{rank} A = \dim \operatorname{Col} A \leq \dim \mathbb{R}^m = m.$$

- On the other hand,

$$\operatorname{rank} A \overset{!}{=} \operatorname{rank}(A^T) = \dim \operatorname{Col}(A^T) \leq \dim \mathbb{R}^n = n.$$

Hence: $\boxed{\operatorname{rank} A \leq \min(m, n).}$

This bound is best possible:

- Suppose that $m \leq n$. Then

$$\begin{array}{cc} & \begin{array}{cc} \overbrace{\hphantom{1\ \ \ddots\ \ 0}}^{m} & \overbrace{\hphantom{0}}^{n-m} \end{array} \\ m\left\{\vphantom{\begin{array}{c}1\\ \ddots \\ 0\end{array}}\right. & \left[\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & 0 \\ 0 & & 1 & \end{array}\right] \end{array}$$

has rank m.

- Suppose that $n \leq m$. Then

$$\begin{array}{cc} \overbrace{\hphantom{1\ \ \ddots\ \ 0}}^{n} & \\ \left[\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \hline & 0 & \end{array}\right] & \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \ddots \\ 0\end{array}}\right\} n \\ \left.\vphantom{0}\right\} m-n \end{array} \end{array}$$

has rank n.

Problem (from 2018/19 exam paper):

- What is the largest possible rank of a 4×7 matrix?

Soln: $\min(4, 7) = 4$

- What is the largest possible rank of a 7×4 matrix?

<u>Soln:</u> $\min(7,4) = 4$

- If the null space of a 4×7 matrix A is 3-dimensional, what is the dimension of its column space?

<u>Soln:</u> Rank-Nullity Thm: $\underbrace{\operatorname{rank} A}_{=\dim \operatorname{Col} A} + \underbrace{\operatorname{nullity} A}_{3} = \underbrace{\#\text{columns of } A}_{7}$

Hence: $\dim \operatorname{Col} A = 7-3 = 4.$

<u>Back to abstract vector spaces:</u>

- Using bases and coordinate vectors, we essentially reduced the study of finitely generated (i.e. finite-dimensional) vector spaces to that of $\mathbb{R}^n$!
- Can we similarly reduce the study of linear transformations to that of matrices? YES, here's how this works.

<u>Setup:</u>
Let V and W be vector spaces with bases $\mathcal{B} = (b_1, \dots, b_n)$ and $\mathcal{C} = (c_1, \dots, c_m)$, respectively. Let $T: V \to W$ be a linear transformation.

Let $F: V \to \mathbb{R}^n,\ v \mapsto [v]_{\mathcal{B}}$ and

$$G: W \to \mathbb{R}^m,\ w \mapsto [w]_{\mathcal{C}}$$

be the coordinate mappings. (Recall that these are isomorphisms)

Then $G \circ T \circ F^{-1}$ is a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$.

$$\mathbb{R}^m \xleftarrow{G} W \xleftarrow{T} V \xleftarrow{F^{-1}} \mathbb{R}^n$$

Therefore, as we saw before, ($\to$ lecture 6) there exists a unique $m \times n$ matrix A s.t. the linear transformation $G \circ T \circ F^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is given by $x \mapsto Ax$. This matrix A is called the matrix of T w.r.t. the bases $\mathcal{B}$ and $\mathcal{C}$ of V and W, respectively.

Explicitly:
$$\mathcal{M}_{\mathcal{C} \leftarrow \mathcal{B}}(T) := A \overset{!}{=} \Big[[T(b_1)]_{\mathcal{C}} \cdots [T(b_n)]_{\mathcal{C}} \Big].$$

The operation $T \rightsquigarrow \mathcal{M}_{\mathcal{C} \leftarrow \mathcal{B}}(T)$ reduces (essentially) everything about linear transformations between finite-dimensional vector spaces to problems involving matrices. Ex: "compute the kernel" becomes "compute the null space".

Ex: Consider the linear transformation $\mathcal{D} : \mathbb{P}_3 \to \mathbb{P}_2$, $\mathcal{D}(p(t)) = p'(t)$.
Choose bases $\mathcal{B} = (1, t, t^2, t^3)$ and $\mathcal{C} = (1, t, t^2)$ of $\mathbb{P}_3$ and $\mathbb{P}_2$, respectively.

Table:

$p(t)$	$D(p(t)) = p'(t)$	$[p'(t)]_{\mathcal{C}}$
1	0	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
t	1	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
t^2	$2t$	$\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$
t^3	$3t^2$	$\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

Hence:
$$\boxed{\mathcal{M}_{\mathcal{C} \leftarrow \mathcal{B}}(\mathcal{D}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.}$$