Applications to matrices

Let A be an $m \times n$ matrix.

Def^n
- The rank of A is $\boxed{\text{rank}\, A := \dim \text{Col}\, A.}$
- The nullity of A is $\boxed{\text{nullity}\, A := \dim \text{Nul}\, A.}$

We saw: if p denotes the number of pivot columns of A, then

$$\boxed{\text{rank}\, A = p} \quad \text{and} \quad \boxed{\text{nullity}\, A = n - \underbrace{p}_{}}$$

$= \#$ free variables

Hence:

Rank-Nullity Theorem: $\boxed{\text{rank}\, A + \text{nullity}\, A = n} = \#$ columns.

In particular, rank A and nullity A determine each other.

This is one of the most fundamental and important results of linear algebra!

Minor (abstract) generalization:

If $T: V \to W$ is a linear transformation between vector spaces, then

$$\boxed{\dim \text{Ker}\, T + \dim \text{Ran}\, T = \dim V.}$$

<u>Invertible Matrix Theorem:</u> Let A be an $n\times n$ matrix.

TFAE: ① A is invertible, i.e. $\exists B$ s.t. $AB = BA = I_n$.

② $\operatorname{rank} A = n$.

③ $\operatorname{nullity} A = 0$.

④ The columns of A form a basis of $\mathbb{R}^n$.

(This follows e.g. by combining the Rank-Nullity Thm and the Basis Thm.)

<u>Rows vs columns</u> Let A be an $m\times n$ matrix. Recall that the <u>transpose</u> of A is the $n\times m$ matrix

$$A^T$$

with $\boxed{(j,i) \text{ entry of } A^T = (i,j) \text{ entry of } A.}$

<u>Ex:</u> $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$

Another pillar of Linear Algebra, somewhat deeper than the Rank-Nullity Thm:

<u>Thm:</u> $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

Cor.: Neither elementary row nor elementary column operations change the rank of a matrix.

(Because elementary column operations don't change the column space, hence don't change the rank. Elementary row operations don't change the rank (of the transpose) either.)

In particular, $\boxed{\text{rank } A = \#\text{ pivot columns in the reduced row echelon form of } A.}$

Ex (of Thm):

$$A := \begin{bmatrix} 3 & 0 & -1 \\ 3 & 0 & -1 \\ 4 & 0 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 0 & -1 \\ 4 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1/3 \\ 1 & 0 & 5/4 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 0 & 19/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ so rank } A = 2$$

$$A^T = \begin{bmatrix} 3 & 3 & 4 \\ 0 & 0 & 0 \\ -1 & -1 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -5 \\ 0 & 0 & 19 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ so rank}(A^T) = 2.$$