The story so far: bases and isomorphisms Let V and W be vector spaces:

- An isomorphism between V and W is a linear transformation $V \to W$ which is one-to-one and onto. We say that V and W are isomorphic iff there exists an isomorphism $V \to W$.
- If $T: V \to W$ is an isomorphism, then so is $T^{-1}: W \to V$. Hence, "V is isomorphic to W" and "W is isomorphic to V" mean the same thing!
- Every vector space is isomorphic to itself: $id_V : V \to V$, $x \mapsto x$ is an isomorphism.
- Slogan: isomorphisms preserve all "linear properties".

  Ex: $(b_1, \dots, b_n)$ is a basis of V

  $\iff (T(b_1), \dots, T(b_n))$ is a basis of W,

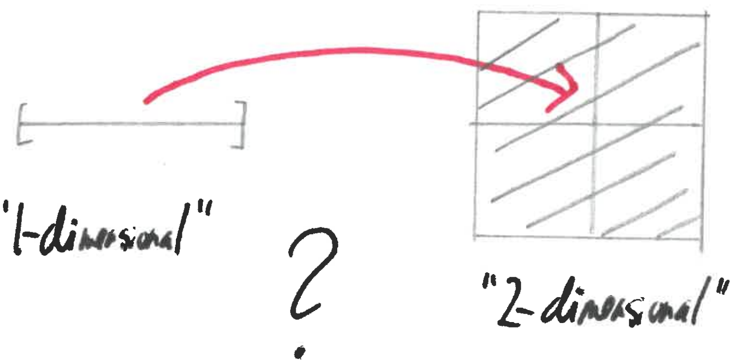
  where $T: V \to W$ is an isomorphism and $b_1, \dots, b_n \in V$.
- Every finitely generated vector space has a basis. More precisely, if $V = \text{span}\{v_1, \dots, v_p\}$, then some of the v_i form a basis of V.
- Given a basis $\mathcal{B} = (b_1, \dots, b_n)$ of V, the coordinate mapping $V \to \mathbb{R}^n$, $x \mapsto [x]_{\mathcal{B}}$ is an isomorphism.

Hence: every finitely generated vector space is isomorphic to ("looks like") $\mathbb{R}^n$ for some n. Q: Can there be more than one such value?

Dimension

In some parts of mathematics, the concept of "dimension" can be difficult and subtle... and even counterintuitive.

Ex. (from analysis) There exists a continuous function from $[0,1]$ onto $[0,1]^2$.



→ Space-filling curves

Fortunately, such issues don't arise in linear algebra!

What do we want?

- Assign a dimension $\dim V$ to each vector space V.
- If V and W are isomorphic, then we want that $\dim V = \dim W$.
- $\dim(\mathbb{R}^n) = n$.
- If $(b_1, \dots, b_n)$ is a basis of V, then $\dim V = n$. (*)

Q: Can we just take (*) as the definition of $\dim V$, at least when V is finitely generated? Does this even make sense?

IOW: Do any two bases of V contain the same number of elements?

Thm: Let $(b_1, \dots, b_n)$ be a basis of V. Then every sequence consisting of (at least) $n+1$ vectors in V is linearly dependent.

Why?
- Using coordinate vectors, we may assume that $V = \mathbb{R}^n$.
- Given $w_1, \dots, w_{n+1} \in \mathbb{R}^n$, consider the linear equation

$$x_1 w_1 + \dots + x_{n+1} w_{n+1} = \underbrace{[w_1 \cdots w_{n+1}]}_{\text{an } n\times(n+1) \text{ matrix}} \begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} = 0.$$

- MA203: There exists a non-trivial soln!

  IOW: $w_1, \dots, w_{n+1}$ are linearly dependent.

Cor.: If V has some basis consisting of n vectors, then every basis of V consists of (precisely) n vectors.

Indeed, if $(b_1, \dots, b_n)$ and $(b_1', \dots, b_m')$ are bases of V, then, by the Thm, $m \leq n \leq m$ and so $m = n$.

Defn: The dimension of V is

$$\dim V = \begin{cases} 0, & \text{if } V = \{0\}, \\ n, & \text{if } V \text{ has a basis } (b_1, \dots, b_n), \\ \infty, & \text{if } V \text{ is not finitely generated.} \end{cases}$$

Fact: Isomorphic vector spaces have the same dimension.

Ex: • $\dim(\mathbb{R}^n) = n$: the standard basis consists of n vectors

• $\dim(\mathbb{P}_n) = n+1$: $(1, t, t^2, \ldots, t^n)$ is a basis

• $\dim(\mathbb{P}) = \infty$ because this space is not finitely generated

How are the concepts of subspaces and dimension related?

Ex: The subspaces of $\mathbb{R}^3$, sorted by dimension, are:

- 0-dim subspaces: just $\{0\}$
- 1-dim —"—: subspaces spanned by a single non-zero vector = lines through the origin
- 2-dim —"—: planes containing the origin
- 3-dim —"—: just $\mathbb{R}^3$. Indeed, any 3 linearly independent vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$. ($\rightarrow$ MA203)

Thm: Let V be a finitely generated vector space and let H be a subspace. Then H is also finitely generated and $\dim H \leq \dim V$. Moreover, any linearly independent sequence of vectors in H can be extended to a basis of V. Finally, $\dim H = \dim V$ iff $H = V$.

Basis Thm: Let $n := \dim V$ satisfy $1 \leq n < \infty$. Let $v_1, \ldots, v_n \in V$. TFAE:
① $(v_1, \ldots, v_n)$ is a basis of V. ② $v_1, \ldots, v_n$ are lin. indep. ③ $V = \operatorname{span}\{v_1, \ldots, v_n\}$.

(Sketch of pf: ① ⇒ ② ③: clear. ③ ⇒ ①: Some of $v_1, \ldots, v_n$ form a basis & every basis contains n vectors. ② ⇒ ①: Extend $v_1, \ldots, v_n$ to basis with n elements.)