Ex: Consider the matrix

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is already in reduced row echelon form.

We observe:
- Non-pivot columns are linear combinations of pivot columns.
- Pivot columns are linearly independent.

Hence: $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)$ is a basis of Col B.

Ex: Let $A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$. Easy: the reduced row echelon form of A is B.

pivot columns of A

Hence: $\left(\begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right)$ is a basis of Col A.

Thm: The pivot columns of a matrix A form a basis of Col A.

Q: Why should we care about bases?
What can we do with them?

Slogan: Each choice of a basis of a vector space provides us with a "coordinate system."

More formally, a choice of a basis $(b_1, \dots, b_n)$ of a vector space V will allow us to identify V and $\mathbb{R}^n$ using the following.

Unique Representation Theorem: Let $(b_1, \dots, b_n)$ be a basis of V. For each $x \in V$, there exists a unique sequence $c_1, \dots, c_n \in \mathbb{R}$ s.t.

$$\boxed{x = c_1 b_1 + \dots + c_n b_n.} \qquad (*)$$

Note: The existence of such $c_1, \dots, c_n$ follows since $V = \operatorname{span}\{b_1, \dots, b_n\}$ and the uniqueness then follows since $b_1, \dots, b_n$ are linearly independent.

Defn Let $\mathcal{B} = (b_1, \dots, b_n)$ be a basis of V. The coordinate vector of $x \in V$ relative to $\mathcal{B}$ is

$$\boxed{[x]_{\mathcal{B}} := \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},}$$

where $c_1, \dots, c_n$ are as in $(*)$.

The function
$$V \longrightarrow \mathbb{R}^n, \quad x \longmapsto [x]_{\mathcal{B}}$$
is the coordinate mapping determined by $\mathcal{B}$.

Ex: let $\mathcal{B} = \left(\underset{e_1}{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \ldots, \underset{e_n}{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}} \right)$ be the standard basis

of $\mathbb{R}^n$. Then $[x]_{\mathcal{B}} = x$ for each $x \in \mathbb{R}^n$.

(Indeed, $x = x_1 e_1 + \ldots + x_n e_n$ so $[x]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x$.)

Hence, taking coordinate vectors generalizes extracting the components of a vector in $\mathbb{R}^n$.

Ex: let $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$.

① $\mathcal{B}$ is a basis of $\mathbb{R}^2$:

- linear independence: Yes, because $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq c \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c \\ 2c \end{bmatrix} \text{ for any } c \in \mathbb{R}.$$

- $\mathcal{B}$ spans $\mathbb{R}^2$: let $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ be arbitrary. We seek to find $c, d \in \mathbb{R}$ with $\begin{bmatrix} x \\ y \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c+d \\ 2d \end{bmatrix}$.
If such numbers exist, then $\boxed{d = y/2.}$

Since $x = c + d = c + y/2$, we then obtain

$$\boxed{c = x - y/2.}$$

Easy: these values for c, d actually work!

② The coordinate mapping determined by $\mathcal{B}$:

We just saw that

$$\left[\begin{bmatrix} x \\ y \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} x - y/2 \\ y/2 \end{bmatrix} \overset{!}{=} \underbrace{\begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}}_{\det = 1/2 \neq 0} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus, the coordinate mapping $\mathbb{R}^2 \to \mathbb{R}^2$, $x \mapsto [x]_{\mathcal{B}}$ is an invertible linear transformation.

= one-to-one & onto

Thm: Let V be a vector space with basis $\mathcal{B} = (b_1, \dots, b_n)$. Then the coordinate mapping $V \to \mathbb{R}^n$, $x \mapsto [x]_{\mathcal{B}}$ is an invertible linear transformation.

Morally, V and $\mathbb{R}^n$ are "essentially the same" in the sense that they have the "same" vector space structure.