Lec 9

Polar Form of Complex Numbers — Powers and Roots.

We can substantially increase the usefulness of the complex plane if besides the x|y coordinates we also employ the usual polar coordinates r, θ defined by

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta . \end{aligned}\right\} \quad *$$

So, $z = x + iy$ takes polar form

$$\begin{aligned} z &= r\cos\theta + i r\sin\theta \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

r here is the absolute value or modulus of z (we had) $= |z|$

So $\underset{\leftarrow \text{New} \rightarrow}{|z| = r} = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$

We said $|z|$ (or r) is the distance from z to the origin.
(similarly $|z_1 - z_2|$ is the distance between z_1 and z_2.)

θ is called the <u>argument</u> of z.
It is denoted by $\arg(z)$.

Dividing the 2^{nd} equation in $*$ by the 1^{st}, we get

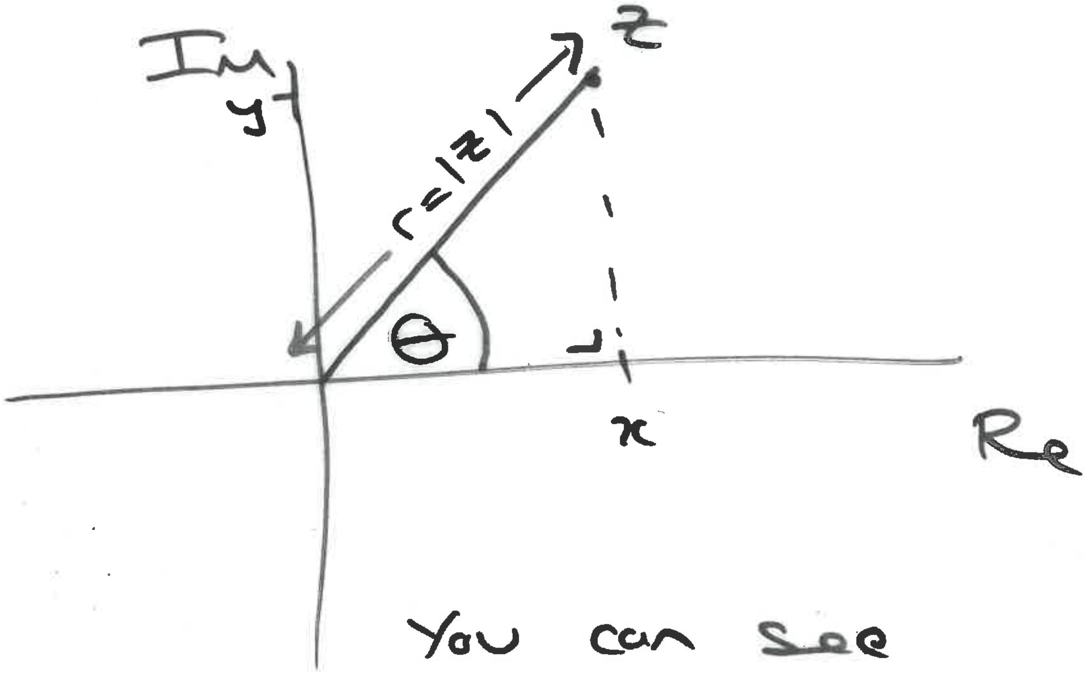
$$\frac{y}{x} = \frac{r \sin\theta}{r \cos\theta}$$

or $\quad \dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta}$

or $\quad \dfrac{y}{x} = \tan\theta \qquad \left[\tan\theta = \dfrac{y}{x}\right]$

or $\quad \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

Geometrically, we have



You can see

$$\tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{y}{x}.$$

All angles are measured in radian and positive in the counterclockwise sense.

Ex $\quad z = 1 + i$

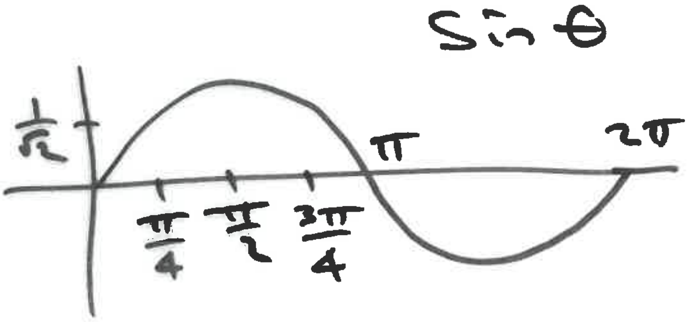
First: find r.

$$r = \sqrt{x^2+y^2} = \sqrt{1^2+1^2} = \sqrt{2}$$

So $$z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right)$$

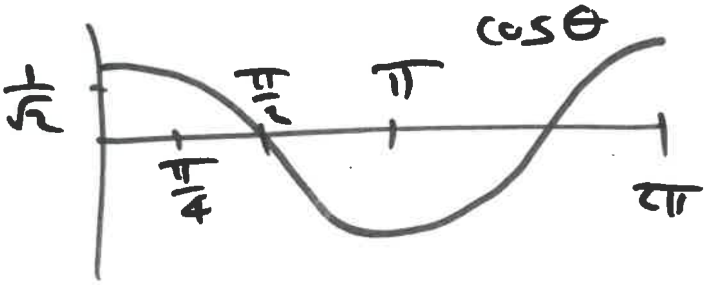
This means $\frac{1}{\sqrt{2}}$ is $\cos\theta$ and

$\frac{1}{\sqrt{2}}$ is $\sin\theta$



$\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

<u>appears</u> to work



$\theta = \frac{\pi}{4}$ or $\frac{7\pi}{4}$

<u>appears</u> to work

$\theta = \frac{\pi}{4}$ is the only value (between 0 and 2π) that satisfies both requirements.

Writing $z = 1 + i$ in polar coordinates, we have

$$z = \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right)\right)$$

Note, in the previous example

we can add or subtract

any integer multiple of 2π

to θ.

This is because the functions

$\sin\theta$ and $\cos\theta$ are

periodic in 2π.

e.g. (from the last example)

$$z = \sqrt{2}\left[\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right]$$

equals

$$\sqrt{2}\left[\cos\left(\tfrac{\pi}{4} + n(2\pi)\right) + i\sin\left(\tfrac{\pi}{4} + n(2\pi)\right)\right]$$

$$n \in \mathbb{Z}$$

Definition

The principal value of the

argument of z $(\neq 0)$

is denoted $\mathrm{Arg}(z)$, with capital "A"

$\theta = \mathrm{Arg}(z)$ satisfies:

$$-\pi < \operatorname{Arg}(z) \leq \pi .$$

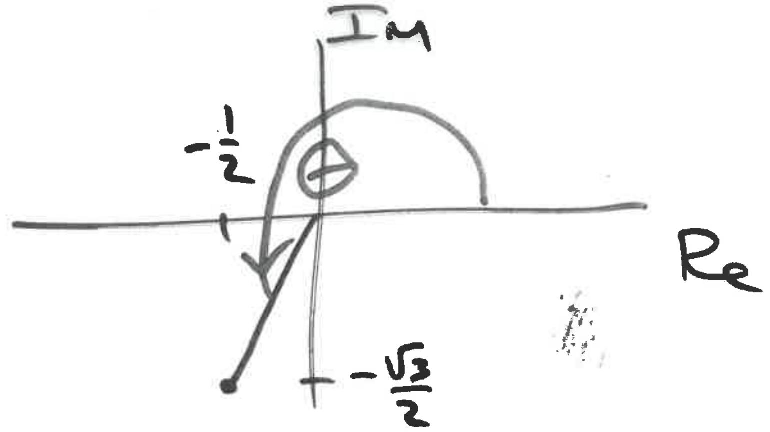
[Practical tip:
when calculating $\operatorname{Arg}(z)$,
find θ such that $0 \leq \theta < 2\pi$
(as in the previous example).
If $\theta > \pi$, just subtract 2π
from you answer].

Ex

Express $z = -3 - 3\sqrt{3}\, i$
(a) in polar form
(b) Find $\operatorname{Arg}(z)$.

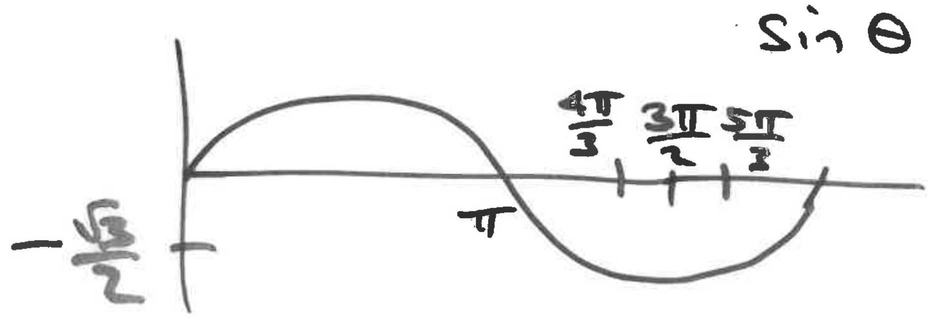
Sol
(a) $r = |z| = \sqrt{(-3)^2 + (-3\sqrt{3})^2}$
$= \sqrt{9+27} = \sqrt{36} = +6$

$$\Rightarrow z = 6\left(\underbrace{-\frac{1}{2}}_{} - \underbrace{\frac{\sqrt{3}}{2}}_{} i\right)$$
$$= 6(\cos\theta + i \sin\theta)$$



From the diagram $\theta = \frac{4\pi}{3}$

OR



$\theta = \frac{4\pi}{3}$ or $\frac{5\pi}{3}$

$\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

The angle that satisfies both requirements is: $\theta = \frac{4\pi}{3}$

So in polar form

$$z = 6\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$$

(b) As $-\pi < \mathrm{Arg}(z) \leq \pi$

$$\text{Arg}(z) \neq \frac{4\pi}{3} > \pi.$$

$$\text{Arg}(z) = \frac{4\pi}{3} - 2\pi = -\frac{2\pi}{3}$$

Caution

when using $\arg(z) = \text{TAN}^{-1}\frac{y}{x}$,

one has to be careful!

e.g if $z_1 = 1+i$, $z_2 = -1-i$

$$\arg(z_1) = \text{TAN}^{-1}\frac{1}{1} = \text{TAN}^{-1} 1 = \frac{\pi}{4}$$

$$\arg(z_2) = \text{TAN}^{-1}\frac{-1}{-1} = \text{TAN}^{-1} 1 = \frac{\pi}{4} \text{ wrong}$$

However,

Therefore,

when using the formula

$\arg(z) = \text{Tan}^{-1}\frac{y}{x}$ and if x or $y < 0$

justify your answer geometrically

For any complex numbers

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

This is known as the triangle inequality.

The generalized triangle inequality

is: $|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|.$

Ex

If $z_1 = 1 + i$

$z_2 = -2 + 3i$

$$|z_1 + z_2| = |-1 + 4i| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$$

$$|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$|z_2| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$$

$$\sqrt{17} < \sqrt{2} + \sqrt{13} \quad \checkmark$$