Lec 6

Ex Is $f(z) = z^3$ analytic?

$$z^3 = (x+iy)^3$$
$$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$
$$= x^3 + (3x^2y)i - 3xy^2 - (y^3)i$$
$$= [x^3 - 3xy^2] + [3x^2y - y^3]i$$

As $f(z) = u(x,y) + i\, v(x,y)$

$$u(x,y) = x^3 - 3xy^2$$
$$v(x,y) = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -3x(2y) = -6xy$$

$$\frac{\partial v}{\partial x} = 6xy$$

$$\therefore \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The Cauchy – Riemann equations are satisfied $\forall z \in \mathbb{C}$

Therefore, $f(z)$ is analytic over all $\mathbb{C}$

Ex

Use the Cauchy – Riemann equations to find all the points $z = x + iy$ where

$$f(z) = x^2 - 2xy - y + i(x + x^2 + y^2)$$

is differentiable

Sol

$$u(x,y) = x^2 - 2xy - y$$

$$v(x,y) = x + x^2 + y^2$$

$$\frac{\partial u}{\partial x} = 2x - 2y$$

$$\frac{\partial v}{\partial y} = 2y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

or $\quad 2x - 2y = 2y$

$\Rightarrow \quad 2x = 4y$

$\Rightarrow \quad y = \frac{1}{2}x$

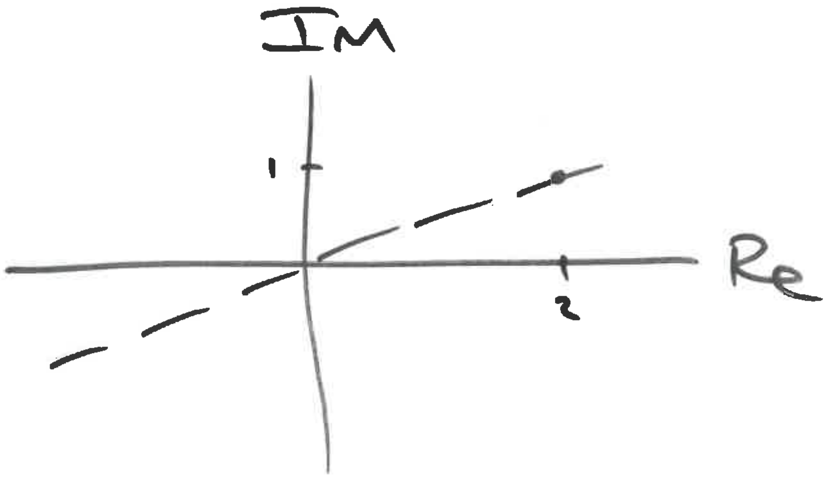
The other C-R equation,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial y} = -2x-1, \quad \frac{\partial v}{\partial x} = 1+2x.$$

$\Rightarrow \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ is satisfied.

Therefore, $f(z)$ is differentiable along the line $y = \frac{1}{2}x$ in the complex plane.

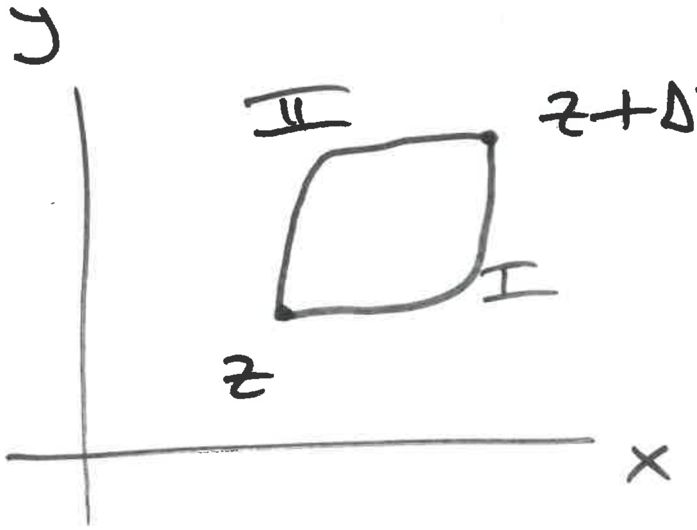


Question:
How do we compute $f'(z)$ where it is differentiable?

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad *$$

In the case of a real variable (Δx) it can only go to zero in two ways.

Δz can go to zero many ways i.e.



Writing $*$ out in full, we get

$$f'(z) = \lim_{\Delta z \to 0} \frac{\left[u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y)\right] - \left[u(x,y) + i v(x,y)\right]}{\Delta x + i \Delta y}$$

Moving along path I

i.e. Let $\Delta y \to 0$ first $(\Delta y = 0)$

and then $\Delta x \to 0$.

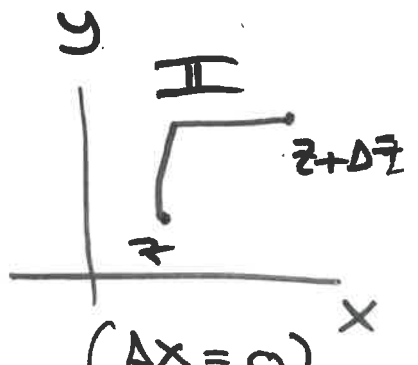
$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} +$$

$$i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

Note: the above limits are partial derifvatives

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad (1)$$

Moving along Path II



i.e. Let $\Delta x \to 0$ first $(\Delta x = 0)$

and then $\Delta y \to 0$

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y}$$

$$+ i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

or $f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ (2)

Equating the real and imaginary parts of (1) and (2) we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

i.e. the C–R equations.

One could use (1) or (2) to compute $f'(z)$.

Ex Find $f'(z)$ for the previous example:

$[f(z) = x^2 - 2xy - y + i(x + x^2 + y^2)]$

Using (1) i.e. $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

So $f'(z) = [2x - 2y] + i[1 + 2x]$

Recall: The derivative only exists along $y = \frac{1}{2}x$ in this Ex

Ex (a) At what points (if any) is the function

$$f(z) = (z - \bar{z}) \operatorname{Re}(z) + z \operatorname{Im}(z)$$

differentiable

(b) Find $f'(z)$ for these z.

Sol (a)

$$z = x + iy$$
$$\bar{z} = x - iy$$
$$z - \bar{z} = iy - (-iy)$$
$$= 2iy$$

$$\operatorname{Re}(z) = x$$
$$\operatorname{Im}(z) = y$$

So
$$f(z) = (2iy)x + (x+iy)y$$
$$= (2xy)i + xy + iy^2$$
$$= [xy] + i[2xy + y^2]$$

$$u(x,y) = xy$$
$$v(x,y) = 2xy + y^2$$

$$\frac{\partial u}{\partial x} = y \qquad \frac{\partial v}{\partial y} = 2x + 2y$$

$$\frac{\partial u}{\partial y} = x \qquad \frac{\partial v}{\partial x} = 2y$$

So $y = 2x + 2y$ Ⓐ $\left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right)$

and $x = -2y$ Ⓑ $\left(\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right)$

must hold.

From Ⓐ $y = -2x$

From Ⓑ $y = -\frac{1}{2}x$

or $-2x = -\frac{1}{2}x$

or $-\frac{3}{2}x = 0 \Rightarrow x = 0$

and $y = 0$

$\therefore$ $f(z)$ is differentiable at $x = 0$ and $y = 0$

i.e. $0 + 0i$ or $z = 0$.

(b) $f'(0)$?

Using $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\Rightarrow f'(z) = [y] + i[2y]$

$\Rightarrow f'(0+0i) = 0 + 0i \quad (= 0).$

Lec $\boxed{7}$

Laplace's equation

If $f(z) = u(x,y) + i\,v(x,y)$

is analytic in a domain D,

then u and v satisfy

Laplace's equation

i.e. $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ①

and $\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ ②

[Notation,

we write $\frac{\partial u}{\partial x} = u_x$, $\frac{\partial^2 u}{\partial y^2} = u_{yy}$

etc.