Lec 24

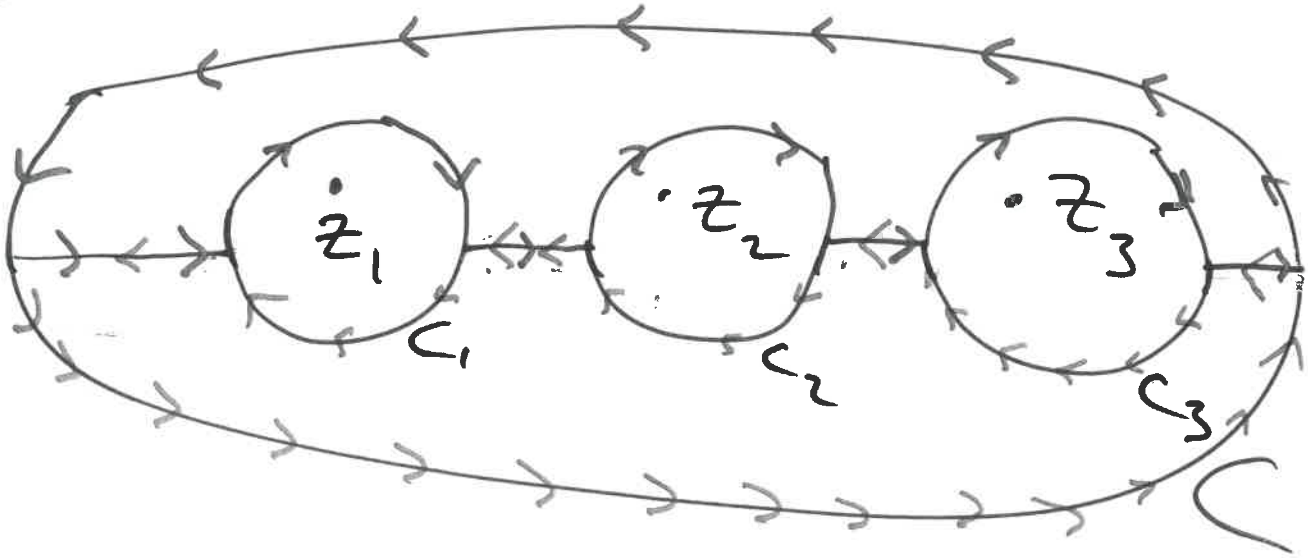
Residue theorem

- Let $f(z)$ be analytic
- inside and on a simple closed curve C
- except for a finitely many singular points $z_1, z_2, \ldots, z_K$ inside C

then

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{K} \operatorname*{Res}_{z=z_j} f(z)$$

Proof



From Cauchys Integral Theorem

$$\oint_C f(z)\,dz = 0 \quad \text{AND} \quad \oint_C f(z)\,dz = 0$$

or
$$\oint_C f(z)\,dz + \oint_C f(z)\,dz = 0$$

$$\Rightarrow \oint_C f(z)\,dz - \oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz - \oint_{C_3} f(z)\,dz =$$

$$\Rightarrow \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz$$

$$\oint_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_1} f(z) + 2\pi i \operatorname*{Res}_{z=z_2} f(z) + 2\pi i \operatorname*{Res}_{z=z_3} f(z)$$

$$\therefore \boxed{\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res} f(z_j)}$$

Ex
$$\oint_C \frac{1}{z(z^2+4)(z-i)^2}\,dz$$

C is $|z - i| = \frac{3}{2}$

The singularities of the integrand are:

a pole of order 1 at $z=0$

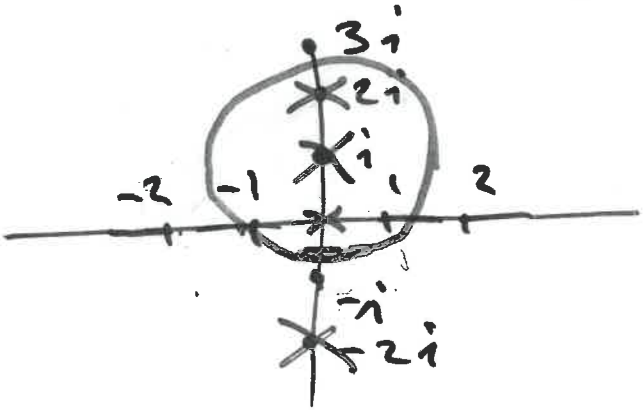
a pole of order 1 at $z=2$

a pole of order 1 at $z=-2$

3 "simple poles"

a pole of order 2 at $z=i$

"Double pole"



$z=0$, $z=i$, $z=2i$ are inside C

and $z=-2i$ is outside C.

$$\operatorname*{Res}_{z=0} \frac{1}{z(z^2+4)(z-i)^2}$$

$$= \lim_{z\to 0} \frac{1}{z(z^2+4)(z-i)^2} \cdot z$$

$$= \lim_{z\to 0} \frac{1}{(z^2+4)(z-i)^2}$$

$$= \frac{1}{4(-i)^2} = \frac{1}{4i^2} = \boxed{-\frac{1}{4}}$$

$$\operatorname*{Res}_{z=2i} \frac{1}{z(z^2+4)(z-i)^2}$$

$$= \operatorname*{Res}_{z=2i} \frac{1}{z(z+2i)(z-2i)(z-i)^2}$$

$$= \lim_{z \to 2i} \frac{1}{z(z+2i)(z-2i)(z-i)^2} \cdot (z-2i)$$

$$= \lim_{z \to 2i} \frac{1}{z(z+2i)(z-i)^2}$$

$$= \frac{1}{2i(2i+2i)(2i-i)^2}$$

$$= \frac{1}{2i(4i)\,i^2} = \boxed{\frac{1}{8}}$$

$$\operatorname*{Res}_{z=i} \frac{1}{z(z^2+4)(z-i)^2}$$

$$= \lim_{z \to i} \left[\frac{1}{z(z^2+4)(z-i)^2} \cdot (z-i)^2 \right]'$$

$$= \lim_{z \to i} \left[\frac{1}{z(z^2+4)} \right]'$$

$$= \lim_{z \to i} \frac{d}{dz}(z^3+4z)^{-1}$$

$$= -(z^3+4z)^{-2}(3z^2+4)\Big|_{z=i}$$

$$= -(i^3+4i)^{-2}(3i^2+4)$$

$$= -\frac{-3+4}{(-i+4i)^2}$$

$$= \frac{-1}{-9} = \boxed{\frac{1}{9}}$$

$$\text{ANSWER} = 2\pi i\left[-\frac{1}{4}+\frac{1}{8}+\frac{1}{9}\right] = \frac{-\pi}{36}i$$

$$\doteq [0]+i[-.09]$$