Lec 22

Residues and the Residue Theorem

From Cauchy's integral theorem $\oint_C f(z)\,dz = 0$

if $f(z)$ is <u>analytic</u> in a simply connected domain D which contains C (a simple closed curve).

<u>If</u> $f(z)$ has a singularity at $z = z_0$ inside C

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$+ \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots \ast$$

Recall $$\oint_{|z|=1} \frac{dz}{z} = 2\pi i$$

<u>Show</u> (using the same method)

$$\oint_{|z-z_0|=R} \frac{dz}{z-z_0} = 2\pi i$$

<u>Show</u> (using the same parameterisation)

$$\oint_{|z-z_0|=R} \frac{dz}{(z-z_0)^k} = 0$$

$$\text{if } k \in \mathbb{Z} \setminus \{+1\}.$$

Integrating around z_0

$$\oint_C f(z)\,dz = \sum_{n=0}^{\infty} a_n \oint_C (z-z_0)^n\,dz$$

$$+ \; b_1 \oint_C \frac{dz}{z-z_0} + b_2 \oint_C \frac{dz}{(z-z_0)^2} + \cdots$$

$$\Rightarrow \oint_C f(z)\,dz = b_1 (2\pi i)$$

<u>NB</u>

$f(z)$ has a singularity at z_0 inside C

Recall : b_1 is the coefficient of $\frac{1}{z-z_0}$ term.

It is called the <u>residue of $f(z)$ at $z=z_0$</u>

<u>Ex</u>

$$\oint_{|z|=5} \frac{\sin z}{z^4}\, dz$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{z}\frac{1}{3!} + \frac{z}{5!} - \cdots$$

So $b_1 = \frac{-1}{3!} = \frac{-1}{6}$

As $|z| = 5$ contains $z = 0$ (the singular point of $\frac{\sin z}{z^4}$)

$$\oint_{|z|=5} \frac{\sin z}{z^4}\, dz = \left(\frac{-1}{6}\right)(2\pi i)$$
$$= -\frac{\pi}{3} i$$

Lec $\boxed{23}$

In the last example ,

we found the FULL Laurent

Series

but we only needed ONE

coefficient i.e. of $\frac{1}{(z-z_0)^1}$ $(b_1$

If the pole has order 1

$$f(z) = \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$\Rightarrow \quad f(z)(z-z_0) = b_1 + a_0(z-z_0) + a_1(z-z_0)^2 + \cdots$$

$$\Rightarrow \quad \boxed{\lim_{z \to z_0} \left[f(z)(z-z_0) \right] = b_1}$$