$$\text{Answer} = \frac{2\pi i}{2}[-18]$$

$$= -18\pi i$$

$$\doteq [0] + [-56.5] i$$

Lec 20

Recall the Taylor series centred at $z = c$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (z-c)^n$$

Ex

Given the Taylor series centred around $z=0$ for $f(z) = e^z$ is:

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

Write out the Taylor series for $f(z) = z^3 e^{z^2}$ with centre $z=0$.

One could use the traditional technique or noting

$$e^{z^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (z^2)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot z^{2n}$$

$$\Rightarrow z^3 e^{z^3} = z^3 \sum_{n=0}^{\infty} \frac{1}{n!} z^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^{2n+3}$$

Laurent Series.

Recall:

The Taylor series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$$

$$= a_0 + a_1(z-c) + a_2(z-c)^2 + \dots$$

Note all the powers are non-negative

The Laurent series is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-c)^n}$$

$\sum_{n=1}^{\infty} \frac{b_n}{(z-c)^n}$ is known as the principal part

There are formulae to calculate a_n and b_n,

however, like in the previous example

there normally is an easier way

<u>Ex</u> (using MacLaurin / Taylor series)

Find the Laurent series for $f(z) = z^{-5}\sin z$ with centre 0.

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

$$\Rightarrow z^{-5}\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4}$$

$$= z^{-5}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots\right]$$

$$= \frac{1}{z^4} - \frac{1}{z^2\,3!} + \frac{1}{5!} - \frac{z^2}{7!} + \frac{z^4}{9!} - \cdots$$

Note: $b_2 = \frac{1}{3!} = \frac{1}{6}$

$b_4 = 1$

$b_n = 0$ if $n \neq 2$ AND $n \neq 4$.

This series holds when $|z| > 0$

We say $f(z)$ has a pole of order 4

Ex (substitution / Taylor series)

Find the Laurent series of $z^2 e^{\frac{1}{z}}$ with centre 0.

Sol

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\Rightarrow e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots$$

$$\Rightarrow z^2 e^{\frac{1}{z}} = z^2 + z + \frac{1}{2!} + \frac{1}{z}\frac{1}{3!}$$

$$+ \frac{1}{z^2}\frac{1}{4!} + \cdots$$

$$\left(= \sum_{n=0}^{\infty} \frac{z^2}{z^n n!} = \sum_{n=0}^{\infty} \frac{1}{z^{n-2} n!} \right)$$

This series holds when $|z| > 0$

Recall :

$$\boxed{NB} \quad \sum_{n=0}^{\infty} w^n = \frac{1}{1-w} \qquad |w|<1$$

Ex

Develop $\frac{1}{1-z}$ in

(a) non-negative powers of z

(b) negative powers of z

Where in the complex plane are the series in (a) and (b) valid?

(a) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{when } |z|<1$$

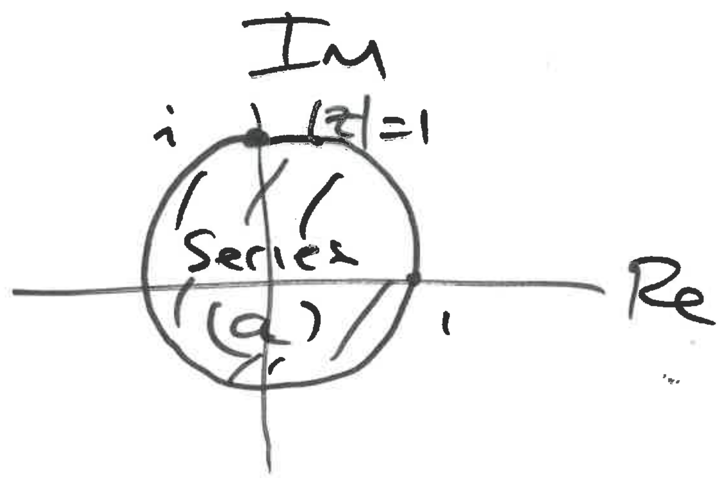
This is because

$$\sum_{n=0}^{\infty} z^n = \lim_{n\to\infty} \frac{1-z^n}{1-z} = \lim_{n\to\infty} \frac{1-r^n e^{in\theta}}{1-re^{i\theta}}$$

if $r>1$, the would equal ∞

so $r<1$ or $|z|<1$.

So



Obviously there is no series for $\frac{1}{1-z}$ when $z=1$

what about when $|z|>1$?

(b) $\frac{1}{1-z}$

$$= \frac{1}{z}\left(\frac{1}{\frac{1}{z}-1}\right) \quad NB$$

$$= -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right)$$

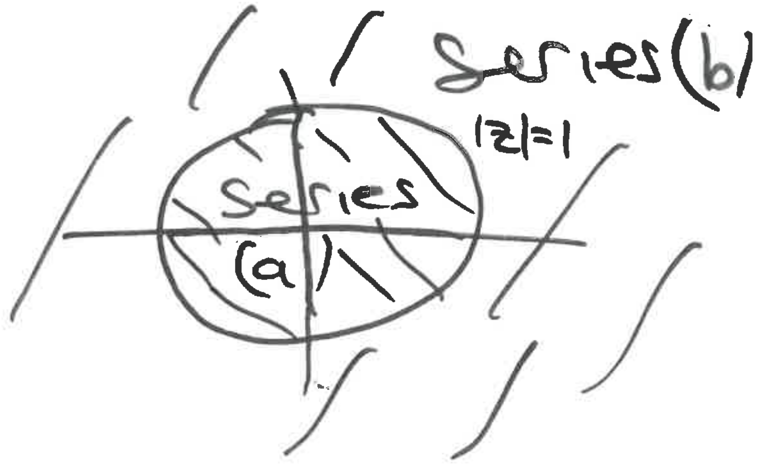
this can be written as

$$-\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right]$$

when $\left|\frac{1}{z}\right|<1$ or $|z|>1$

or $-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}-\cdots$

To sum up



<u>Ex</u>

$$f(z) = \frac{1}{z^3 - z^4}$$

$$= \frac{1}{z^3}\left[\frac{1}{1-z}\right]$$

So when $\quad 0 < |z| < 1$

$$\frac{1}{z^3}\left[\text{Answer in (a) above}\right]$$

$$= \frac{1}{z^3}\left[\sum_{n=0}^{\infty} z^n\right]$$

$$= \frac{1}{z^3}\left[1 + z + z^2 + \cdots\right]$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \ldots$$

The is a pole at $z=0$ of order 3.

when $|z| > 1$

$$\frac{1}{z^3}\left[\text{Answer in (b) above}\right]$$

$$= \frac{1}{z^3}\left[-\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots\right]$$

$$= -\frac{1}{z^4} - \frac{1}{z^5} - \frac{1}{z^6} - \cdots$$

__Ex__ Expand

$$f(z) = \frac{z}{z-2}$$

in a Laurent series

with centre 0

when (a) $|z| < 2$ (b) $|z| > 2$

(a) Aim to get $f(z)$ in

the form $\dfrac{1}{1-*}$

$$f(z) = z\left[\frac{1}{z-2}\right]$$

$$= -z\left[\frac{1}{2-z}\right]$$

$$= -\frac{z}{2}\left[\frac{1}{1-z/2}\right]$$

Note this can be expanded in the usual way if $|z/2|<1$

i.e. $|z|<2$

i.e.

$$-\frac{z}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right]$$

$$= -\frac{z}{2}-\frac{z^2}{4}-\frac{z^3}{8}-\cdots$$

(b)

$$\frac{z}{z-2} = \frac{1}{z}\left[\frac{z}{1-2/z}\right]$$

$$= \frac{1}{1-\frac{2}{z}}$$

Note this can be expanded in the usual way if $\left|\frac{2}{z}\right|<1$

i.e. $|z|>2$

So $f(z)$

$$= 1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{z^n}$$