Lec

19 Cauchy's integral formula for derivatives.

If – $f(z)$ is analytic

– in a simply connected domain D

– and C is a simple closed curve in D

– and C encloses the point z_0

then $$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i\, f^{(n)}(z_0)}{n!}$$

Exercise :

Derive the above by differentiating Cauchy's Integral formula with respect to z_0.

Ex

$$\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz$$

where C is any simple closed curve containing, $z_0=\pi i$

Sol

$$\oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i\, f^{(n)}(z_0)}{n!}$$

Let $n=1$

$$\oint \frac{f(z)}{(z-z_0)^2}\,dz = \frac{2\pi i\, f'(z_0)}{1!}$$

Let $f(z) = \cos z$

$z_0 = \pi i$

$$\Rightarrow f'(z) = -\sin z$$

$$\Rightarrow f'(\pi i) = -\sin(\pi i)$$

$$= -i\sinh(\pi)$$

$[\text{As } \sin(iz) = i\sinh(z)]$

So

$$\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz = \frac{2\pi i}{1!}\left[-i\sinh(\pi)\right]$$

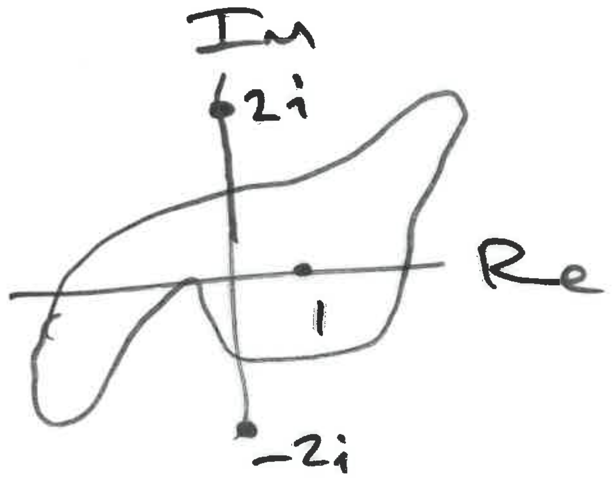
$$= 2\pi\sinh(\pi) \doteq 72.6+0$$

Ex

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}\, dz$$

C is any contour that contains $z=1$ and does not contain $z=2i$ or $z=-2i$.

e.g.



Sol

$$\oint_C \frac{e^z/(z^2+4)}{(z-1)^2}$$

Letting $n=1$, we get

$$\oint_C \frac{f(z)}{(z-z_0)^2}\, dz = 2\pi i\, f'(z_0)$$

Let $z_0 = 1$ and

$$f(z) = \frac{e^z}{z^2+4}$$

$$f'(z) = \frac{(z^2+4)e^z - e^z(2z)}{(z^2+4)^2}$$

$$f'(1) = \frac{5e - e(2)}{25} = \frac{3e}{25}$$

$$\text{Answer} = 2\pi i\left[\frac{3e}{25}\right] = \frac{6\pi e i}{25}$$

$$(\doteq 0 + 2.05 i)$$

EX

$$\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3}\,dz, \quad C \text{ contains } z_0 = -i$$

Letting $n = 2$, we get

$$\oint_C \frac{f(z)}{(z-z_0)^3}\,dz = \frac{2\pi i\, f^{(2)}(z_0)}{2!}$$

here $f(z) = z^4 - 3z^2 + 6$, $z_0 = -i$

$$f'(z) = 4z^3 - 6z$$

$$f^{(2)}(z) = 12z^2 - 6$$

$$f^{(2)}(-i) = 12(-i)^2 - 6$$

$$= -12 - 6 = -18$$

$$\text{Answer} = \frac{2\pi i}{2}\left[-18\right]$$

$$= -18\pi i$$

$$\doteqdot [0] + [-56.5]i$$

$$\text{Answer} = \frac{2\pi i}{2}[-18]$$

$$= -18\pi i$$

$$\doteqdot [0] + [-56.5] i$$

Lec 20

Recall the Taylor Series centred at $z = c$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(z-c)^n$$

Ex

Given the Taylor series centred around $z=0$ for $f(z) = e^z$ is:

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$