(A simply connected region in the complex plane does not have any holes.)

Lec 16

The first integration method:

<u>$f(z)$ has to be analytic</u>

Let $f(z)$ be analytic in a simply connected domain D, then there exists an indefinite integral of $f(z)$ in D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D and for all paths in D joining two points z_0 and z_1 in D

we have

$$\int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0)$$

Note :

we can write z_0 and z_1 instead of C since we get the same value for all those C from z_0 to z_1.

Ex 1

$$\int_0^{1+i} z^2\, dz$$

$$= \left.\frac{z^3}{3}\right|_0^{1+i}$$

$$= \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3} i$$

Ex 2

$$\int_{-\pi i}^{\pi i} \cos z \quad dz$$

$$= \sin z \Big|_{-\pi i}^{\pi i}$$

$$= \sin(\pi i) - \sin(-\pi i)$$

$$= \sin(\pi i) + \sin(\pi i)$$

$$= 2\sin(\pi i)$$

We had $\sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$

so $2\sin(\pi i) = 2\left[\dfrac{e^{i(\pi i)} - e^{-i(\pi i)}}{2i}\right]$

$$= \frac{1}{i}\left[e^{-\pi} - e^{\pi}\right] = -i\left(e^{-\pi} - e^{\pi}\right).$$

$$= i\left(e^{\pi} - e^{-\pi}\right)$$

$$\doteq [0] + i\,[23.1]$$

<u>Ex 3</u> $\displaystyle\int_{8+\pi i}^{8-3\pi i} e^{z/2}\, dz$

$$= 2e^{\frac{z}{2}} \Big|_{8+\pi i}^{8-3\pi i}$$

$$= 2\left[e^{\frac{8-3\pi i}{2}} - e^{\frac{8+\pi i}{2}} \right]$$

$$= 2\left[e^{4-\frac{3}{2}\pi i} - e^{4+\frac{\pi}{2} i} \right]$$

$$= 2e^{4}\left[e^{i(-\frac{3}{2}\pi)} - e^{i(\pi/2)} \right]$$

$$= 2e^{4}\left[\left\{ \cos\left(-\tfrac{3}{2}\pi\right) + i \sin\left(-\tfrac{3}{2}\pi\right) \right\} - \left\{ \cos\left(\pi/2\right) + i \sin(\pi/2) \right\} \right]$$

$$= 0$$

The integrands in the above 3 examples were analytic over the entire complex plane

D could equal the complex plane

Ex 4

$$\int_{-i}^{i} \frac{1}{z}\, dz$$

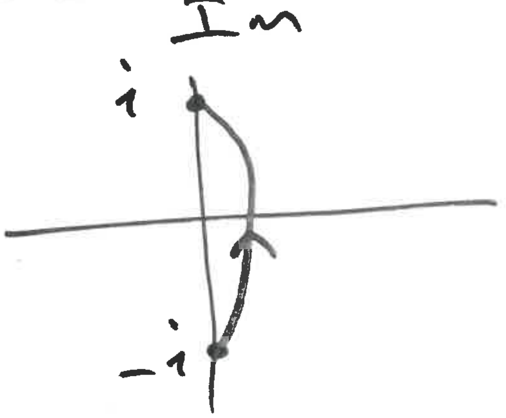
$\frac{1}{z}$ is not analytic at $z=0$

Here, let D equal

$$\left\{ z \in \mathbb{C} \;\middle|\; Re(z) \geq 0 \setminus \{0\} \right\}$$

This is simply connected.

One integration path could be



(we are avoiding $z = 0$).

$$\int_{-i}^{i} \frac{dz}{z} =$$

$$\operatorname{Ln} z \Big|_{-i}^{i} = \operatorname{Ln}(i) - \operatorname{Ln}(-i) *$$

$$\operatorname{Ln} i = \frac{\pi}{2} i$$

Recall!

$$\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) = -i = e^{i\left(-\frac{\pi}{2}\right)}$$

$$\therefore \operatorname{Ln}(-i) = -i\frac{\pi}{2}$$

So $*$ is $\left(\frac{\pi}{2} i\right) - \left(-i \pi/2\right) = i\pi$

Note:

$$z = re^{i\theta}$$

So $\quad \ln z = \ln r + \ln e^{i\theta}$

$$= \ln r + i\theta ,$$

$\therefore \quad \ln z = \ln r + i \arg(z)$

The value of $\ln z$ corresponding to $\operatorname{Arg}(z)$ is denoted $\operatorname{Ln}(z)$ and is called the principal value of $\ln(z)$.

So $\quad \operatorname{Ln}(z) = \ln r + i \operatorname{Arg}(z)$

so $\quad \ln(z) = \operatorname{Ln}(z) + 2n i\pi$

$$n \in \mathbb{Z}.$$

The second integration method:

(This method is not restricted to analytic functions and applies to any continuous complex function.)

- Let C be a piecewise smooth path represented by $z = z(t)$ with $a \leq t \leq b$.

- Let $f(z)$ be a continuous function on C

Then

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\,\dot{z}(t)\,dt$$

Here $\dot{z}(t) = \dfrac{dz}{dt}$.

Steps for the 2^{ND} method are :

(A) Represent the path C in the form $z(t)$ $a \le t \le b$

(B) Calculate $\dot{z}(t)$

(C) Substitute $z(t)$ in for all z in $f(z)$
(i.e. $x(t)$ for x and $y(t)$ for y)

(D) Integrate $f(z(t))\dot{z}(t)$ from a to b.

Ex

$$\oint_{|z|=1} \frac{1}{z}\, dz$$ An important example.

$|z|=1$ is the unit circle, centre 0

Note: We cannot use integration method 1 because to use method 1

" $f(z)$ has to be analytic in a simply connected domain D"

$f(z)$ is not analytic at $z=0$
(Any region containing $|z|=1$ must contain $z=0$)
If this point is removed e.g

D is $\frac{1}{2} < |z| < \frac{3}{2}$

D is <u>not</u> simply connected
(because it has a hole in it.)

<u>Sol</u>

(A) $z(t) = \cos t + i \sin t$

$= e^{it} \quad 0 \leq t \leq 2\pi$

(B) $\dot{z}(t) = i e^{it}$

(C) $f(z(t)) = \frac{1}{z} = \frac{1}{e^{it}}$

$\therefore f(z(t)) = e^{-it}$

(D) $\int_0^{2\pi} [e^{-it}][i e^{it}] \, dt$

$$= \int_0^{2\pi} i \, dt$$

$$= i t \Big|_{t=0}^{t=2\pi} = 2\pi i$$

NB $\boxed{\oint_{|z|=1} \frac{dz}{z} = 2\pi i}$

Note if we incorrectly used method 1

the answer $= 0$.

Ex

Evaluate

$$\int_C (\overline{z}^2 + 1) \, dz$$

where C is a contour joining 0 to $2i$ along

(1) the line segment joining 0 to $2i$ parametrized by

$z(t) = 2it \qquad t \in [0, 1]$

(ii) the semicircle parametrized by

$$z(t) = i + e^{it} \quad t \in [-\pi/2, \pi/2]$$

Sol

$f(z) = \bar{z}^2 + 1$ is not analytic

$\longrightarrow$ use method 2.

(i) $z(t) = [0] + i[2t]$

$$\bar{z}^2 + 1 = (x - iy)^2 + 1$$

$$= ([0] - i[2t])^2 + 1$$

$$= -4t^2 + 1$$

$$\therefore f(z(t)) = -4t^2 + 1$$

$$z(t) = 2it$$

$$\dot{z}(t) = 2i$$

$$\int_C f(z)\,dz = \int_0^1 f(z(t))\,\dot{z}(t)\,dt$$

$$= \int_0^1 [-4t^2 + 1](2i)\,dt$$

$$= (2i)\left[-\frac{4t^3}{3} + t\right]\Bigg|_{t=0}^{t=1}$$

$$= 2i\left(-\frac{4}{3}+1\right) = -\frac{2}{3}i$$

(ii) $$z(t) = i + e^{it} \qquad t \in [-\pi/2, \pi/2]$$

$$z(t) = i + \cos t + i\sin t$$

$$= [\cos t] + i[1+\sin t]$$

$$\overline{z(t)} = [\cos t] - i[1+\sin t]$$

$$= \cos t - i\sin t \quad - i$$

$$= e^{-it} - i$$

$$[\overline{z}(t)]^2 + 1 = (e^{-it} - i)^2 + 1$$

$$\dot{z}(t) = ie^{it}$$

$$\int_C (\overline{z}^2 + 1)\,dz = \int_{-\pi/2}^{\pi/2} [(e^{-it} - i)^2 + 1]\, ie^{it}\, dt$$

$$= \int_{-\pi/2}^{\pi/2} [e^{-2it} - 2ie^{-it} - 1 + 1]\, ie^{it}\, dt$$

$$= i\int_{-\pi/2}^{\pi/2} [e^{-it} - 2i]\, dt$$

$$= \int_{-\pi/2}^{\pi/2} (i e^{-it} + 2)\, dt$$

$$= -e^{-it} + 2t \Big|_{-\pi/2}^{\pi/2}$$

$$\left\{ -\left[\cos(-\pi/2) + i \sin(-\pi/2)\right] + 2(\pi/2) \right\}$$

$$- \left\{ -\left[\cos(+\pi/2) + i \sin(+\pi/2)\right] - 2(\pi/2) \right\}$$

$$= -\cos \pi/2 + i \sin \pi/2 + \pi$$

$$+ \cos \pi/2 + i \sin \pi/2 + \pi$$

$$= 2\pi + 2i$$

Note different answer to $-2/3$

eventhough

in both cases are integrating

from 0 to $2i$.

In the 2^{nD} integration method

$\rightarrow$ path matters.