Lec 14 Complex Powers.

Ex

Write i^i in the form $x+iy$, $x,y \in \mathbb{R}$, and plot the results in the complex plane.

Sol

1) Take the Ln and 2) exp of i^i.

$$\operatorname{Ln} i^i$$

$$= i \operatorname{Ln} i$$

So $i^i = e^{i \operatorname{Ln} i}$ *

Recall: $e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$\Rightarrow \quad i\frac{\pi}{2} = \operatorname{Ln} i$

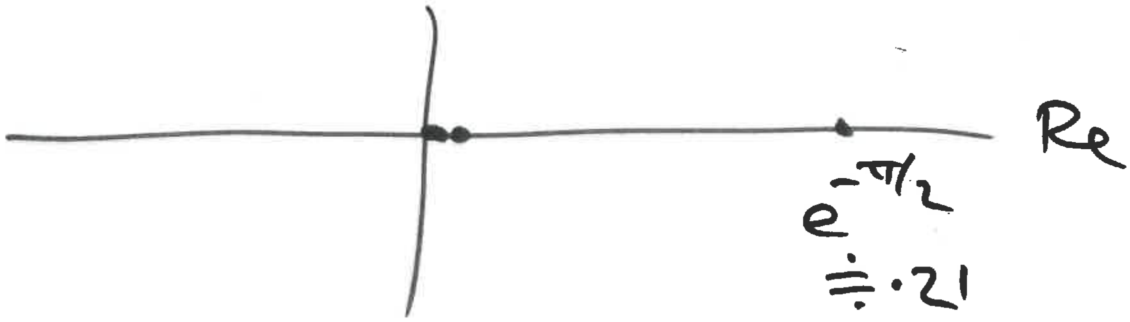
or $\quad i\left(\frac{\pi}{2} + 2n\pi\right) = \operatorname{Ln} i \quad n \in \mathbb{Z}$

[because $e^{i(\pi/2 + 2n\pi)} = \cos\left(\frac{\pi}{2} + 2n\pi\right) + i\sin\left(\frac{\pi}{2} + 2n\pi\right) = i$]

Substituting the above into $\ast$, we get

$$i^{i} = e^{i\left[i\left(\frac{\pi}{2}+2n\pi\right)\right]}$$

$$= e^{-\pi/2 - 2n\pi}$$



Ex

Find all values of

$$(3i)^{i}$$

and express them in the form $x+iy$, $x, y \in \mathbb{R}$.

$$\text{Ln} \Rightarrow \quad \text{Ln}\,(3i)^{i}$$

$$= i\,\text{Ln}(3i)$$

$$\exp \Rightarrow \quad e^{i\,\text{Ln}\,(3i)}$$

So $(3i)^{i} = e^{i\,\text{Ln}(3i)}$

or $(3i)^i = e^{i[\ln 3 + \ln i]}$

$$= e^{i[\ln 3 + i(\frac{\pi}{2} + 2n\pi)]}, \quad n \in \mathbb{Z}$$

$$= e^{-(\frac{\pi}{2} + 2n\pi)} e^{i \ln 3}$$ "exponential polar form"

$$= e^{-(\frac{\pi}{2} + 2n\pi)} \left[\cos(\ln 3) + i \sin(\ln 3)\right]$$

— "polar form"

or

$$= \left[e^{-\pi/2 + 2n\pi} \cos(\ln 3)\right] + i \left[e^{-\pi/2 + 2n\pi} \sin(\ln 3)\right]$$

— "$x + iy$" form $x, y \in \mathbb{R}$

<u>Ex</u>

Find all values of $(i-1)^{2i}$ and express them in polar form.

Sol

$$\text{Ln} \Rightarrow \quad \text{Ln}(i-1)^{2i} = 2i\,\text{Ln}(i-1)$$

$$\exp \Rightarrow \quad (i-1)^{2i} = e^{2i\,\text{Ln}(i-1)}$$

But, $i-1 = -1+i = \sqrt{2}\left[-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right]$

$$= \sqrt{2}\left[\cos\frac{3\pi}{4} + i\,\text{Sin}\,\frac{3\pi}{4}\right]$$

$$= \sqrt{2}\left[\cos\left(\frac{3\pi}{4} + 2n\pi\right) + i\,\text{Sin}\left(\frac{3\pi}{4} + 2n\pi\right)\right], \quad n \in \mathbb{Z}$$

$$= \sqrt{2}\, e^{i\left(\frac{3\pi}{4} + 2n\pi\right)}$$

So $\quad (i-1)^{2i} = e^{2i\,\text{Ln}\left\{\sqrt{2}\, e^{i(3\pi/4 + 2n\pi)}\right\}}$

$$= e^{2i\left[\text{Ln}\sqrt{2} + i(3\pi/4 + 2n\pi)\right]} \qquad n \in \mathbb{Z}$$

$$= e^{2i\,\text{Ln}(\sqrt{2})}\, e^{-2(3\pi/4 + 2n\pi)}$$

$$= e^{i\,\text{Ln}\,2}\, e^{-(3\pi/2 + 4n\pi)}$$

$$= e^{-(3\pi/2 + 4n\pi)}\, e^{i\,\text{Ln}\,2}$$

$$= e^{-3\pi/2 + 4n\pi}\left[\cos(\text{Ln}\,2) + i\,\text{Sin}(\text{Ln}\,2)\right] \quad n \in \mathbb{Z}$$

Ex

Show $\sin^{-1}(z) = -i\,\mathrm{Ln}\left(iz + \sqrt{1-z^2}\right)$.

sol

First use

(and show as an exercise) that

$$\sin(w) = \frac{e^{iw} - e^{-iw}}{2i}, \quad w \in \mathbb{C}$$

and define $z \in \mathbb{C}$ such that

$z = \sin(w)$

or $w = \sin^{-1}(z)$.

Note: We are being asked to find an expression for w.

We can write

$$\sin(w) = \frac{e^{iw} - e^{-iw}}{2i}$$

writing z for $\sin w$, we get

$$z = \frac{e^{iw} - e^{-iw}}{2i}$$

or $$2iz = e^{iw} - e^{-iw},$$

or $$2ize^{iw} = e^{2iw} - 1$$

or $$0 = e^{2iw} - 2ize^{iw} - 1$$

or $$0 = (e^{iw})^2 + (-2iz)(e^{iw}) + (-1)$$

So $$e^{iw} = \frac{2iz + (-4z^2+4)^{1/2}}{2}$$

$$e^{iw} = iz + (-z^2+1)^{1/2}$$

Take the log of both sides

$$iw = \operatorname{Ln}\left(iz + (-z^2+1)^{1/2}\right)$$

or $$w = -i \operatorname{Ln}\left(iz + (-z^2+1)^{1/2}\right)$$

As required.

i.e. $$\sin^{-1} z = -i \operatorname{Ln}\left(iz + (-z^2+1)^{1/2}\right)$$

Ex

Find all the values of

$\sin^{-1}(2i)$.

Sol $\sin^{-1}(z) = -i \,\mathrm{Ln}\left(iz + \sqrt{1-z^2}\right)$

We need to find all the square roots of $1-(2i)^2$

$= 1-(-4) = 5$

$5 = 5\left[\cos 0 + i\sin 0\right] = 5e^{i\cdot 0}$

$= 5\left[\cos(2n\pi) + i\sin(2n\pi)\right] = 5e^{i2n\pi}, n\in\mathbb{Z}$

Taking the square root

$\sqrt{5}\, e^{in\pi} \quad n\in\mathbb{Z}$.

So, from above

$\sin^{-1}(2i) = -i\,\mathrm{Ln}\left(i(2i) + \sqrt{5}e^{in\pi}\right)$

$= -i\,\mathrm{Ln}\left(-2+\sqrt{5}e^{in\pi}\right)$

$$e^{in\pi} = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

$\underline{n \text{ Even}} \iff +\sqrt{5}$

$$\sin^{-1}(2i) = -i\,\mathrm{Ln}(-2+\sqrt{5})$$

$$= -i\,\mathrm{Ln}\left[(-2+\sqrt{5})e^{i2\pi k}\right] \quad k\in\mathbb{Z}$$

writing $-2+\sqrt{5}>0$ in "exponential" polar form"

$$= -i\left[\mathrm{Ln}(\sqrt{5}-2) + \mathrm{Ln}(e^{i2\pi k})\right]$$

$$= -i\left[\mathrm{Ln}(\sqrt{5}-2) + 2\pi k i\right]$$

$$= +2\pi k - i\,\mathrm{Ln}(\sqrt{5}-2) \quad *$$

$\underline{n \text{ odd}} \iff -\sqrt{5}$

$$\sin^{-1}(2i) = -i\,\mathrm{Ln}(-2-\sqrt{5})$$

$$= -i\,\mathrm{Ln}\left([2+\sqrt{5}](-1)\right)$$

$$= -i\,\mathrm{Ln}\left([2+\sqrt{5}]e^{i(2k+1)\pi}\right), \quad k\in\mathbb{Z}$$

writing $-2-\sqrt{5}$ in "exponential polar form"

$$= -i\left[\mathrm{Ln}(2+\sqrt{5}) + i(2k+1)\pi\right]$$

$$= + (2k+1)\pi \quad - i\, \mathrm{Ln}(2+\sqrt{5})$$

Grouping this set of solutions with $*$

$$\left\{ 2\pi k - i\,\mathrm{Ln}(\sqrt{5}-2) \,\middle|\, k \in \mathbb{Z} \right\} \cup \left\{ (2k+1)\pi - i\,\mathrm{Ln}(2+\sqrt{5}) \,\middle|\, k \in \mathbb{Z} \right\}$$

Exercise:

Show and use:

$$\cos w = \frac{1}{2}\left(e^{iw} + e^{-iw} \right)$$

to show that

$$\cos^{-1} z = -i\,\mathrm{Ln}\left[z + \sqrt{z^2-1} \right]$$

Lec 15

Complex Integration

Recall, definite integrals

e.g $\displaystyle\int_1^2 x^2\,dx$