

lec
13

Description of sets of points in the Argand diagram.

Ex A line segment from $-1-i$ to $1+i$.

We parameterize the line segment by introducing a parameter e.g. t .

Pick two arbitrary values of t e.g. 0 and 1.

Find the values of a and b such that

$$z(t) = at + b \quad a, b \in \mathbb{C}$$

$$t=0 \quad -1-i = a(0) + b$$
$$\Rightarrow -1-i = b \quad [= -(1+i)]$$

$$t=1 \quad 1+i = a(1) + b$$

$$\text{or } 1+i = a - 1 - i$$

$$\text{or } 2(1+i) = a$$

So $z(t) = z(1+i)t - (1+i)$

When: $t=0$ we get $-1-i$
 $t=1$ we get $1+i$

[Exercise:

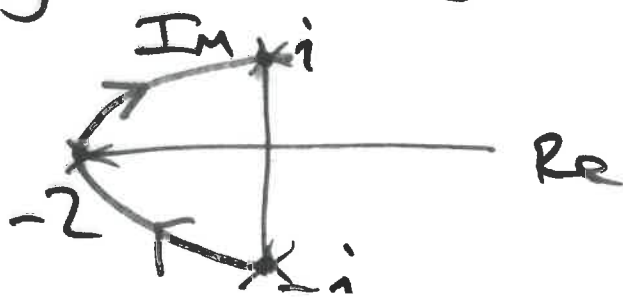
Find the line if you
 pick $t=-1$ and $t=1$]

The full description of
 the line segment is:

$$\left\{ z(t) \in \mathbb{C} \mid z(t) = z(1+i)t - (1+i), \right. \\ \left. t \in [0, 1] \right\}$$

Ex

Describe a parabola
 in the complex plane
 going through the points:



ie starting
 at $-i$ and
 ending at i

$$z(t) = at^2 + bt + c \quad a, b, c \in \mathbb{C}$$

Let $t = -1$ when $z = -i$ $\left[z(-1) = -i \right]$

$$z(0) = -2 \quad \text{and} \quad z(1) = i$$

$$z(-1) = -i \Rightarrow -i = a - b + c, \quad \text{--- (1)}$$

$$z(0) = -2 \Rightarrow \boxed{-2 = c} \quad \text{--- (2)}$$

$$z(1) = i \Rightarrow i = a + b + c. \quad \text{--- (3)}$$

$$\text{(2)} \rightarrow \text{(1)} \Rightarrow -i = a - b - 2 \quad \left. \vphantom{\text{(2)} \rightarrow \text{(1)}} \right\}$$

$$\text{(2)} \rightarrow \text{(3)} \Rightarrow i = a + b - 2 \quad \left. \vphantom{\text{(2)} \rightarrow \text{(3)}} \right\}$$

Add the above 2 equations

$$0 = 2a - 4 \Rightarrow \boxed{a = 2}$$

$$\text{As } i = a + b - 2$$

$$\Rightarrow \boxed{i = b}$$

$$\text{So } z(t) = 2t^2 + it - 2$$

The full description is:

$$\left\{ z \in \mathbb{C} \mid z(t) = 2t^2 + it - 2, \right. \\ \left. t \in [-1, 1] \right\}$$

If the parameterization
has highest power

one: line / line segment

two: parabola

three: cubic polynomial etc

The start and end
parameters will tell
the start and end points

Ex

If given

$$\left\{ z(t) = t^2 - it, \quad t \in [0, 1] \right\}$$

— it is a parabola,

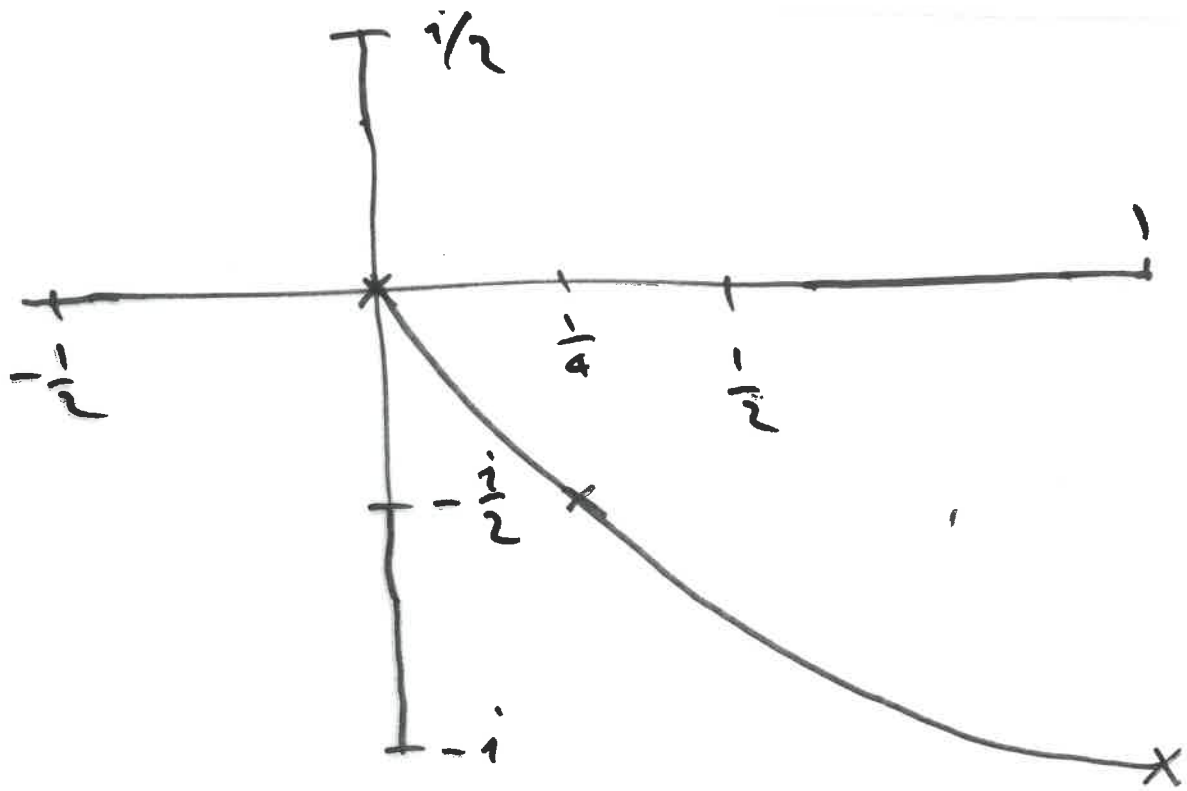
— its starting point is 0

— its ending point is $1 - i$

— it also contains

$$\left(\frac{1}{2}\right)^2 - i\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{i}{2}$$

So



A circle can be described in a number of ways.

First way

Ex $2 = |z - 1 - i|$

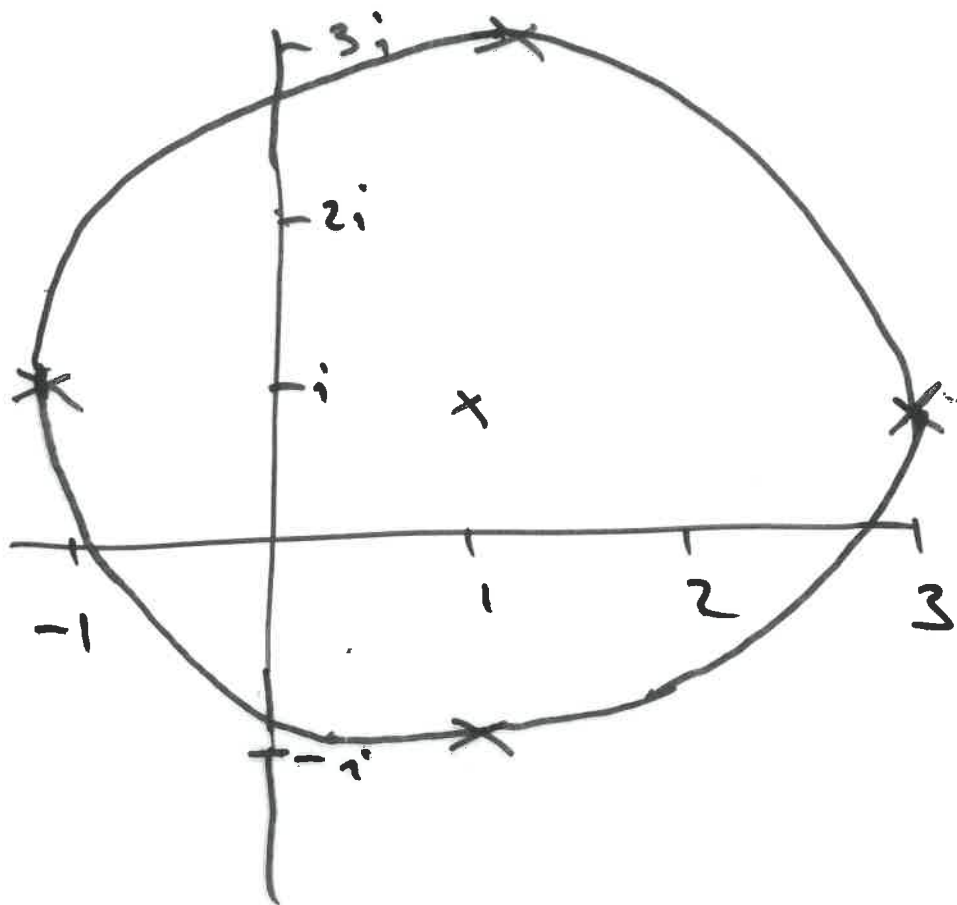
is a circle.

We can write it as

$$2 = |z - (1+i)|$$

The RHS is saying:

"the distance between $1+i$ and z " is 2 units



In general

$$\{ z \in \mathbb{C} \mid |z - a| = r \} \quad r \in \mathbb{R}^+$$

is a circle with centre $a \in \mathbb{C}$
and radius r .

$$\{ z \in \mathbb{C} \mid |z - a| < r \}$$

is the interior of the circle

$$\{ z \in \mathbb{C} \mid |z - a| \leq r \}$$

is the interior of the circle
plus the circle.

This is known as a disc

$$\left\{ z \in \mathbb{C} \mid |z - a| > r \right\}$$

is the exterior of the circle.

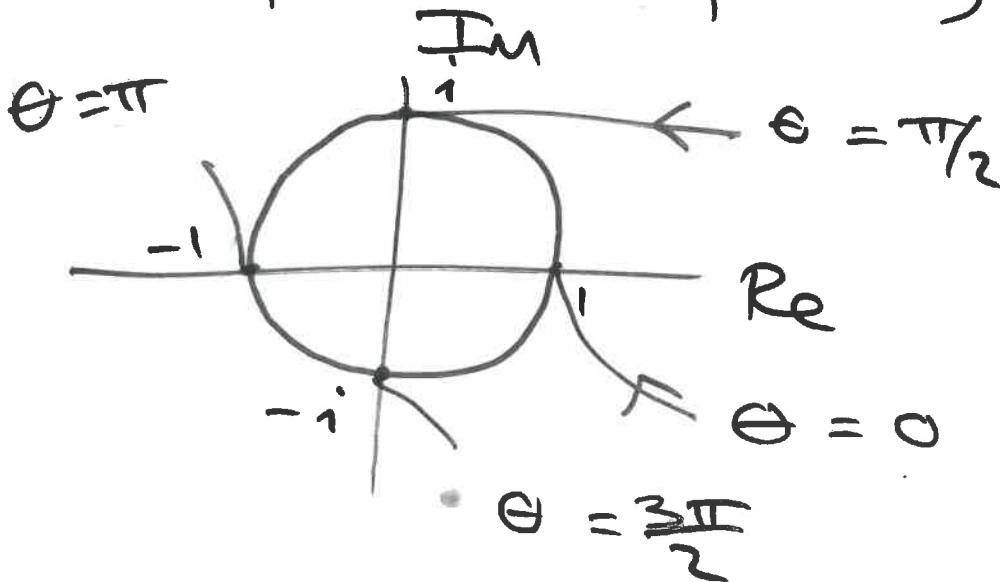
Second way

Note:

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

If one let θ run from 0 to 2π

and plot the points, we get



we get a circle.

Ex

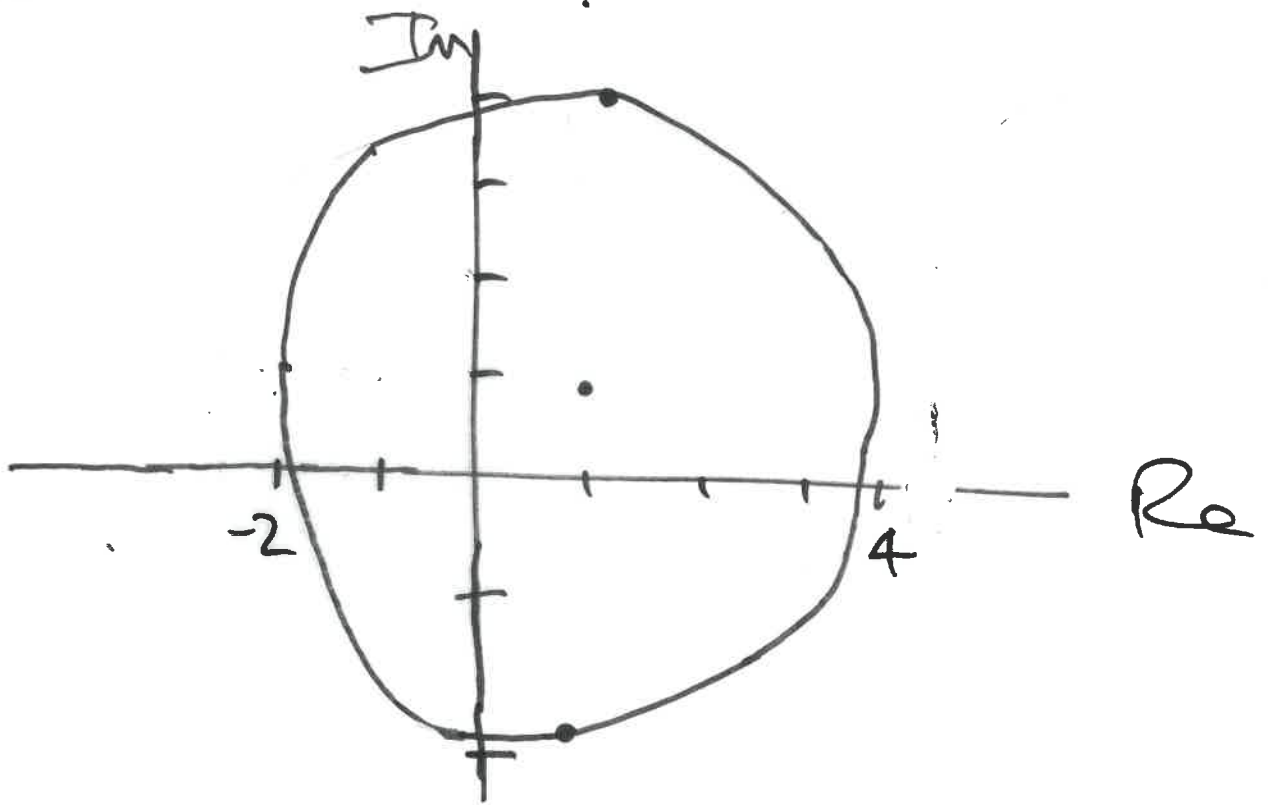
$$ze^{i\theta}$$

$$0 \leq \theta \leq 2\pi$$

is a circle : Centre 0
and radius 2.

Ex
 $z(\theta) = 1 + i + 3e^{i\theta}$

is a circle : Centre $1+i$
and radius 3.

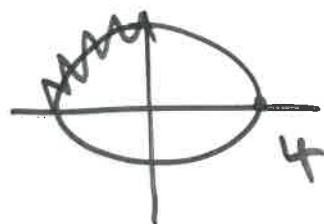


In general a circle of
centre $z_0 \in \mathbb{C}$ with radius $r_0 \in \mathbb{R}^+$
can be parametrised by

$$\left\{ z(\theta) = z_0 + r_0 e^{i\theta}, \quad 0 \leq \theta \leq 2\pi \right\}$$

Ex

Parameterise an arc of a circle of centre 0 and radius 4, that is in the 2nd quadrant.



$$\left\{ z(\theta) = 4e^{i\theta}, \quad \frac{\pi}{2} \leq \theta \leq \pi \right\}$$

Limits

We treat limits in complex analysis in the same way as we treated them in real analysis.

Ex

Evaluate

$$\lim_{z \rightarrow 2i} \frac{z-2i}{z+4}$$

We can factorise the denominator

$$\lim_{z \rightarrow 2i} \frac{z - 2i}{(z - 2i)(z + 2i)}$$

$$= \lim_{z \rightarrow 2i} \frac{1}{z + 2i}$$

$$= \frac{1}{2i + 2i}$$

$$= \frac{1}{4i} = \frac{i}{4i^2} = -\frac{1}{4}i$$

$$= (0) + \left(-\frac{1}{4}\right)i$$

Ex

Evaluate

$$\lim_{z \rightarrow -2i} (-z^5 - iz)$$

$$= -(-2i)^5 - i(-2i)$$

$$= 32i^5 + 2i^2$$

$$= 32(i^4)(i) + 2(-1)$$

$$= (-2) + (32)i$$