Lec

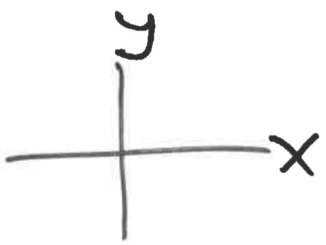
12 Mapping of complex function

In calculus graphs of function $y = f(x)$ in the x/y plane give: – a visual representation

– and lead to a better understanding of the function.

This is similar in complex analysis

In real analysis / calculus we needed just 2 dimensions



for a complex function

$$f(z) = u(x,y) + i\,v(x,y) \qquad (z = x+iy)$$

we need 2 dimensions to plot the x and y i.e. complex plane and we need another 2 dimensions to plot u and V i.e. a second complex plane. This complex plane is called the <u>w plane</u>.

We let $\underline{w = f(z)} = u + iV$.

We say, f defines a <u>mapping</u> of points in the z plane into the w plane.

We say,

$w_0 = f(z_0)$ is <u>the image</u> of z_0 with respect to f.

<u>Ex</u>

Mapping $w = z^2$ i.e. $f(z) = z^2$

In polar coordinates, we have

$$z = r(\cos\theta + i\sin\theta) \quad *$$

Let $f(z) = R(\cos\phi + i\sin\phi)$

From $*$, we have

$$z^2 = r^2(\cos(2\theta) + i\sin(2\theta))$$

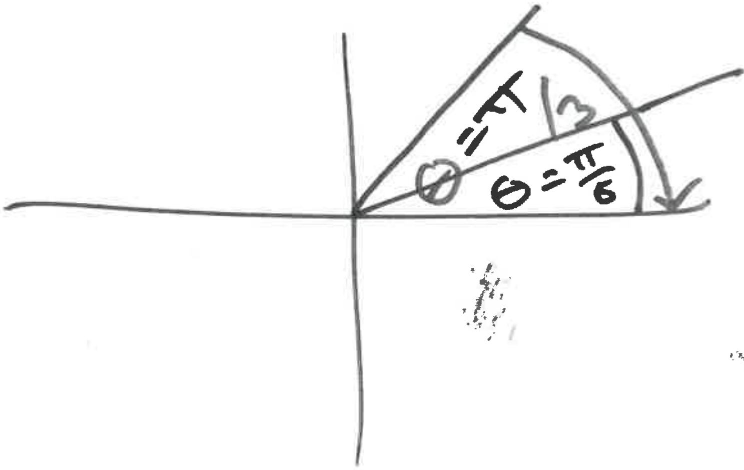
Comparing moduli and arguments, we have

$$R = r^2 \quad \text{and} \quad \phi = 2\theta.$$

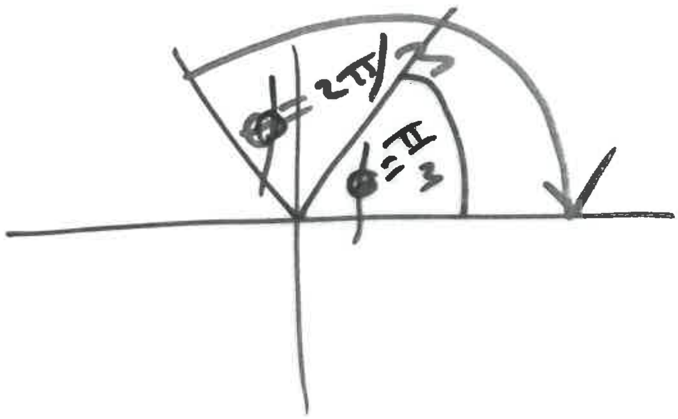
Hence, circles $r = r_0$ are mapped onto circles $R = r_0^2$

Rays $\theta = \theta_0$ are mapped onto $\phi = 2\theta_0$.

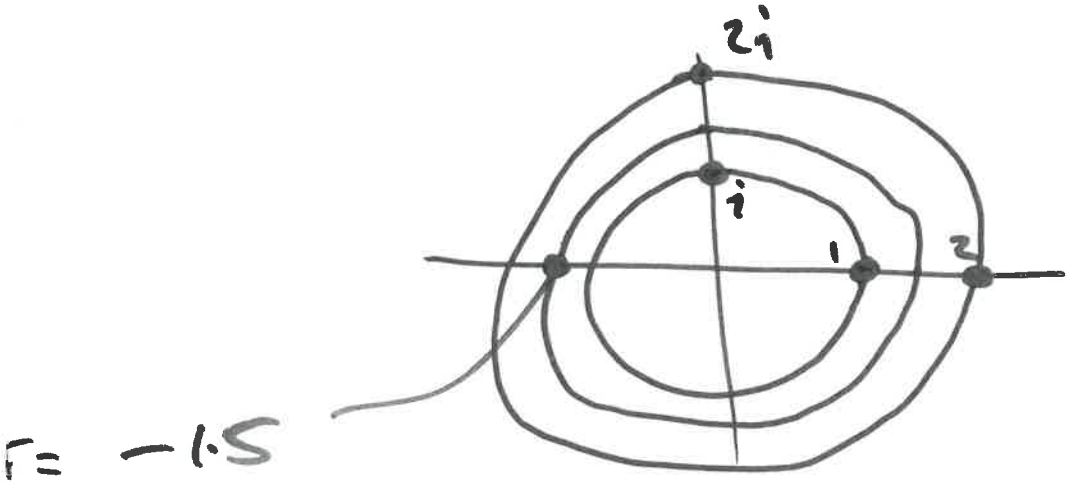
Graphically, we have



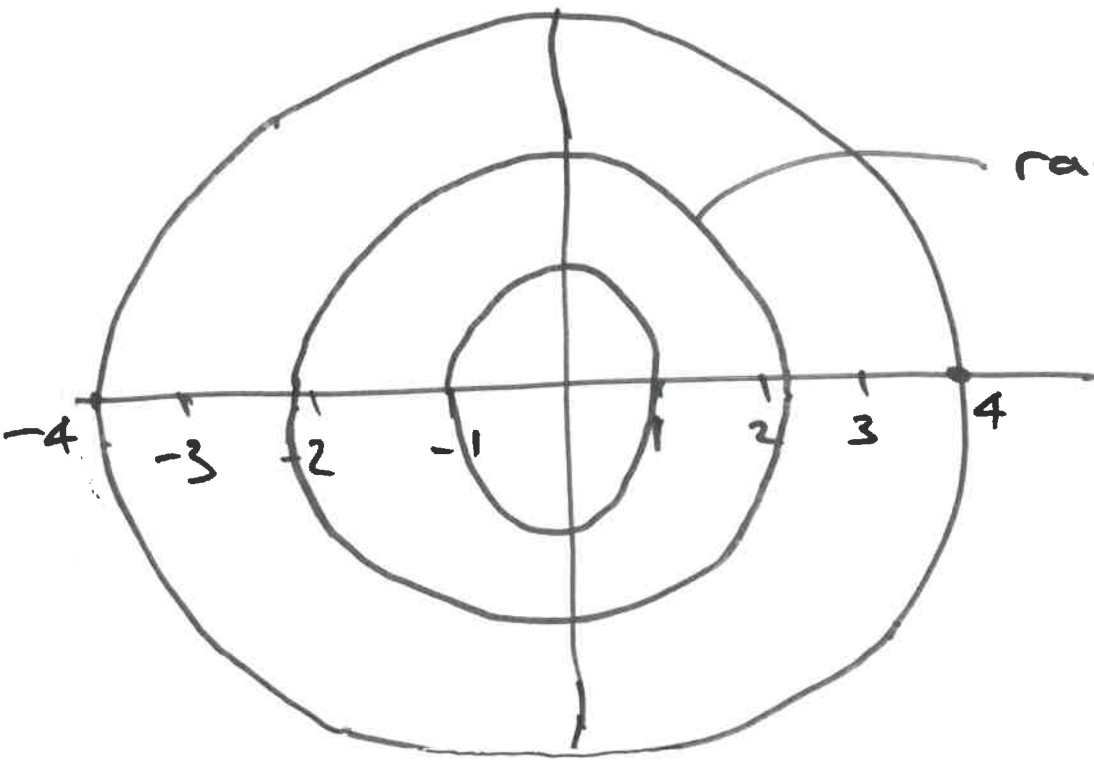
z-plane



w-plane



z-plane



radius 2.25
= R

w-plane

The Exponential Function

The complex exponential function is e^z.

Also written as $\exp(z)$.

$$e^z = e^{x+iy}$$

$$= e^x e^{iy} \quad *$$

$$\boxed{e^{iy} = \cos y + i \sin y.} \quad \underline{NB}$$

The Maclaurin series for

$$e^* = 1 + * + \frac{*^2}{2!} + \frac{*^3}{3!} + \frac{*^4}{4!} + \frac{*^5}{5!} + \cdots$$

So $$e^{iy} = 1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots$$

So $$e^{iy} = 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \frac{iy^5}{5!} + \cdots$$

$$e^{iy} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)$$

or $e^{iy} = \cos(y) + i \sin(y)$ as above

Recall, the polar form of a complex number

$$z = r(\cos\theta + i\sin\theta)$$

This can be written as,

$$z = re^{i\theta}$$ **NB**

because, we showed (with y instead of θ)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Important formulae:

1) $$e^{i(\theta + 2n\pi)} = e^{i\theta} \qquad n \in \mathbb{Z}.$$

because

$$e^{i(\theta+2n\pi)} = \cos(\theta + 2n\pi) + i\sin(\theta+2n\pi)$$
$$= \cos\theta + i\sin\theta = e^{i\theta}.$$

2) $$e^{i(\frac{\pi}{2})} = i$$ **NB**

because

$$e^{i(\pi/2)} = \cos\frac{\pi}{2} + i\sin\pi/2$$
$$= (0) + i(1) = i$$

Exersise : Show

$e^{\pi i} = -1$, $e^{-\pi i/2} = -i$ and $e^{-\pi i} = -1$

Also

$$|e^{iy}| = |\cos y + i \sin y|$$
$$= \sqrt{\cos^2 y + \sin^2 y} = \sqrt{1} = 1$$

So

$$|e^z| = |e^{x+iy}| = |e^x||e^{iy}| = |e^x|$$

Ex

If $z = 2 + 3\pi i$

find e^z and $|e^z|$

$$e^{2+3\pi i} = e^2 e^{3\pi i}$$
$$= e^2 e^{\pi i} = -e^2.$$

So $e^{2+3\pi i} \approx -7.389$ $(-e^2)$

$|e^{2+3\pi i}| \doteq 7.389$ (e^2)

<u>Ex</u> Write $z_1 = 1+i$ and $z_2 = \sqrt{-i}$ in "exponential polar form".

$$z_1 = 1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

$$= \sqrt{2}\, e^{i(\pi/4)}$$

Do z_2 as an exercise.