Lec 11 Roots of a complex number.

As $z = r[\cos\theta + i\sin\theta]$

we can say

$$z = r[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)]$$

with $n \in \mathbb{Z}$

We can add / subtract any integer multiple of 2π to the argument of the cos and sin functions.

This is because these functions are periodic in 2π.

$$\sqrt[k]{z} = \sqrt[k]{r}\left[\cos\left(\frac{\theta + 2n\pi}{k}\right) + i\sin\left(\frac{\theta + 2n\pi}{k}\right)\right]$$

There are up to k k^{th} roots of z.

To find all the roots let $n = 0, 1, 2, \ldots k-1$.

If, we let
$n = k, k+1,$ etc
$\to$ the roots are repeated.

Ex

Find the three cube roots of unity (1).

Sol

$$z = 1$$

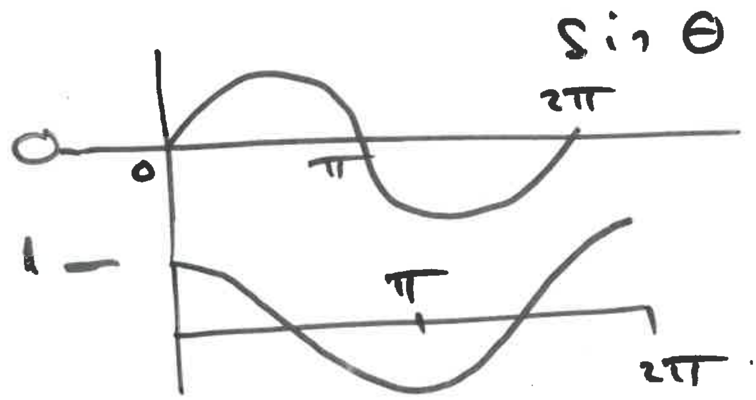
or $z = 1 + 0i$.

We need to write z in polar form i.e.
find: r and θ.

$$x = 1 \qquad y = 0$$

$$r = \sqrt{x^2+y^2} = \sqrt{1^2+0^2} = \sqrt{1} = 1$$

So $z = 1\,(1 + 0i)$



$\theta = 0, \pi$

$\theta = 0, 2\pi$

For both to hold $\theta = 0$.

$$z = 1\left[\cos 0 + i\sin 0\right]$$

$$z = 1\left[\cos 2n\pi + i\sin 2n\pi\right]$$

$$\sqrt[3]{z} = \sqrt[3]{1}\left[\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\right]$$

$$= 1\left[\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\right]$$

$$= \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$$

Let $n=0$, we get:

$$\cos 0 + i\sin 0 = \underline{1.}$$

Let $n=1$, we get:

$$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

$$= \underline{-\frac{1}{2} + i\frac{\sqrt{3}}{2}}$$

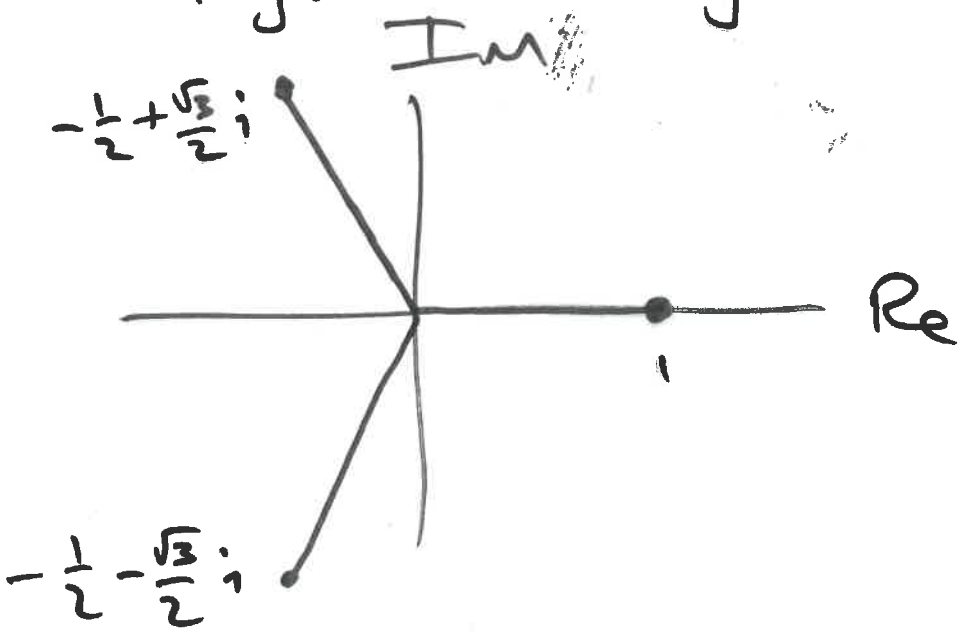
Let $n=2$, we get:

$$\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

$$= \underline{-\frac{1}{2} - i\frac{\sqrt{3}}{2}}$$

Ex

Plot the above roots in the Argand diagram.



Exercise :

Check that:

$$z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i \quad \text{and} \quad z_3 = 1$$

satisfy $z^3 = 1$.

Ex

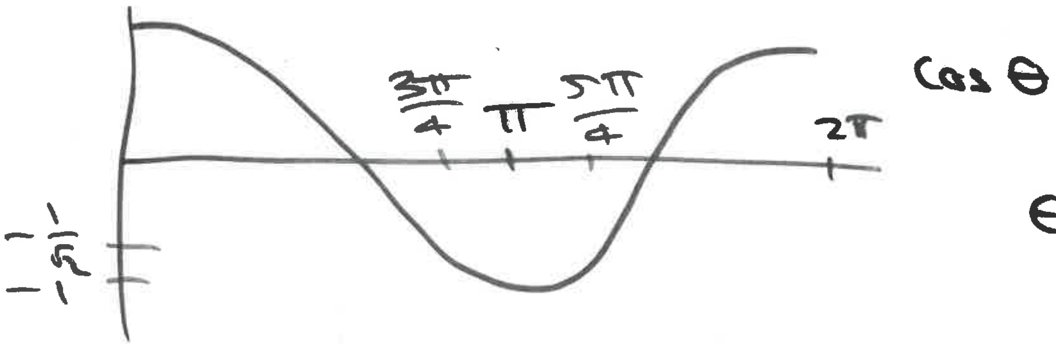
Find all the values of $(4i - 4)^{1/4}$ and locate them graphically in the complex plane.

Sol

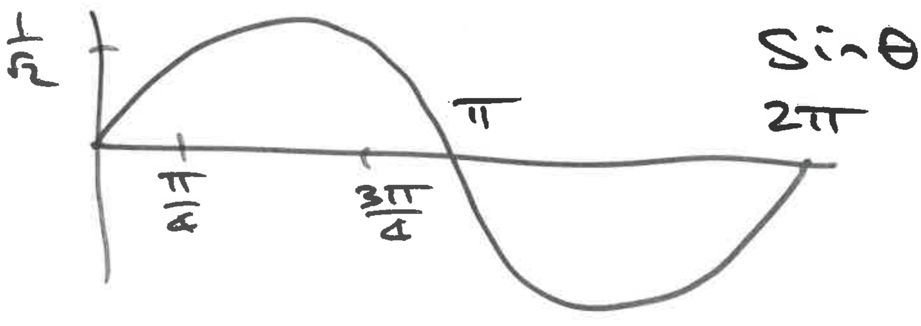
Write $z = 4i - 4$ in polar form.

$$r = |z| = \sqrt{4^2 + (-4)^2}$$
$$= \sqrt{16+16} = \sqrt{32}$$
$$= \sqrt{(2)(16)} = +4\sqrt{2}$$

So
$$z = 4\sqrt{2}\left[\frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}}\right]$$
$$= 4\sqrt{2}\left[\underbrace{-\frac{1}{\sqrt{2}}}_{\cos\theta} + \underbrace{\frac{1}{\sqrt{2}}}_{\sin\theta} i\right]$$



$\theta = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$



$\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

So $\theta = \frac{3\pi}{4}$.

or $z = 4\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$

or $z = 4\sqrt{2}\left[\cos\left(\frac{3\pi}{4} + 2\pi n\right) + i\sin\left(\frac{3\pi}{4} + 2\pi n\right)\right]$

$n \in \mathbb{Z}$

$\Rightarrow z^{\frac{1}{4}} = (4\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{\frac{3\pi}{4} + 2\pi n}{4}\right) + i\sin\left(\frac{\frac{3\pi}{4} + 2\pi n}{4}\right)\right]$

Let $n=0$: $(4\sqrt{2})^{\frac{1}{4}}\left[\cos\frac{3\pi}{16} + i\sin\frac{3\pi}{16}\right] = z_1$

$n=1$: $(4\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{3\pi}{16} + \frac{\pi}{2}\right) + i\sin\left(\frac{3\pi}{16} + \frac{\pi}{2}\right)\right]$

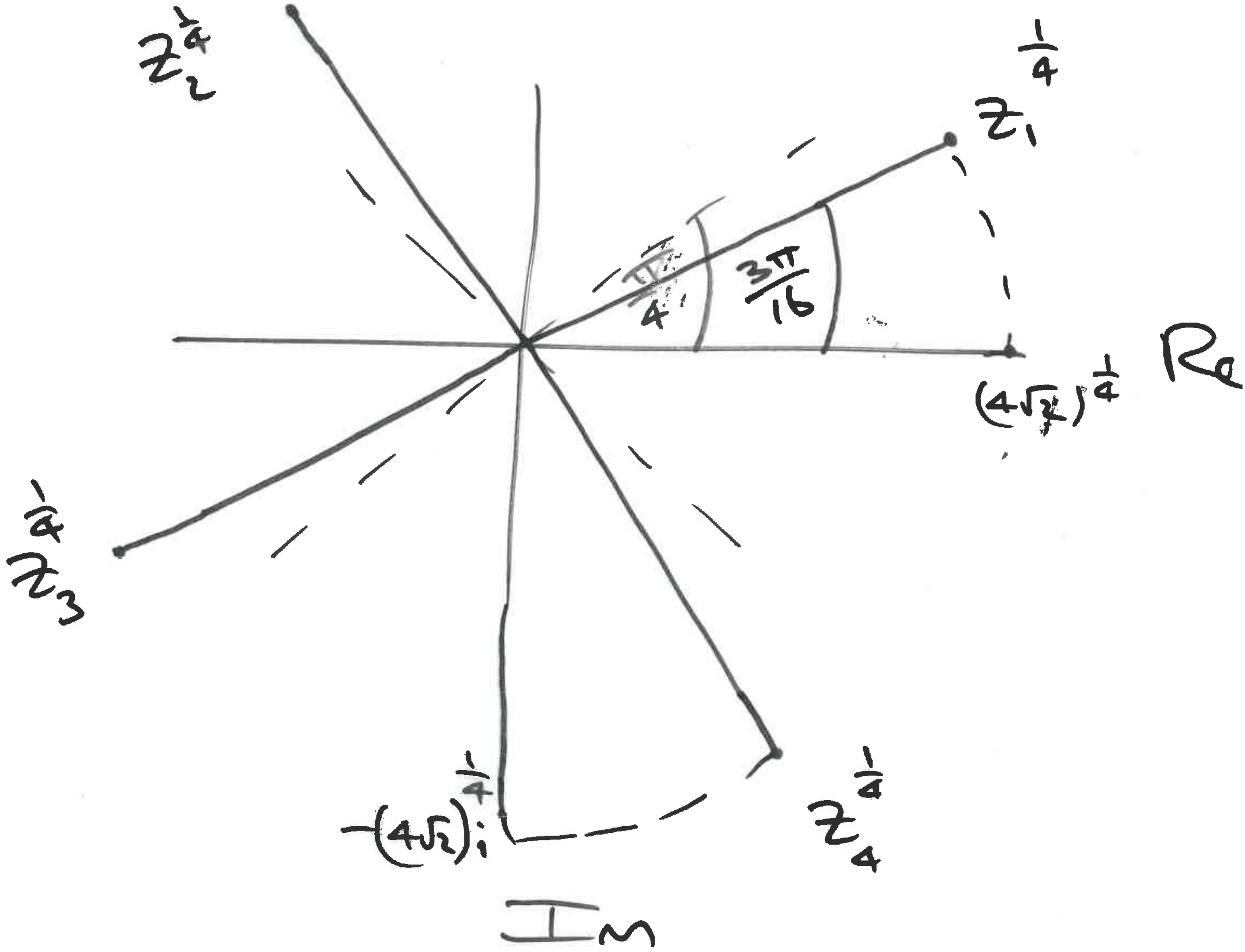
$\therefore z_2 = (4\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{11\pi}{16}\right) + i\sin\left(\frac{11\pi}{16}\right)\right]$

$n=2$: $(4\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{3\pi}{16} + \pi\right) + i\sin\left(\frac{3\pi}{16} + \pi\right)\right]$

$\therefore z_3 = (4\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{19\pi}{16}\right) + i\sin\left(\frac{19\pi}{16}\right)\right]$

$n=3$: $(4\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{3\pi}{16} + \frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{16} + \frac{3\pi}{2}\right)\right]$

$\therefore z_4 = (4\sqrt{2})^{\frac{1}{4}}\left[\cos\left(\frac{27\pi}{16}\right) + i\sin\left(\frac{27\pi}{16}\right)\right]$



We can use complex numbers in polar form to solve certain equations

<u>Ex</u>

Solve for z when

$$z^5 + 243 = 0$$

and plot your answer.

Sol $\quad z^5 = -243 + 0i$

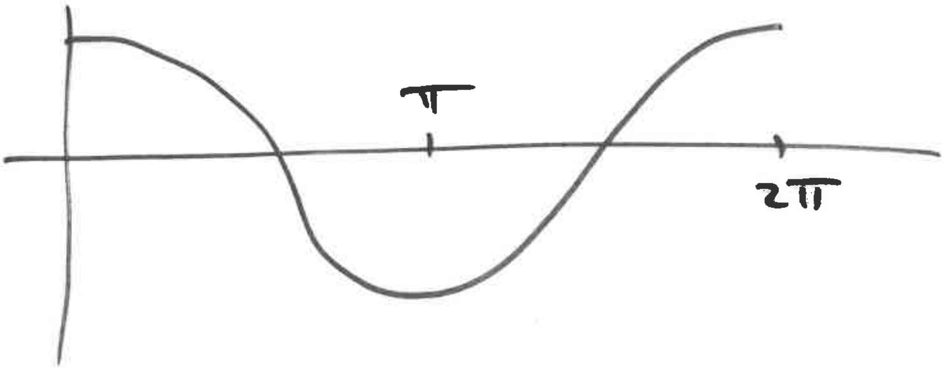
i.e. we need to find:

$z = (-243 + 0i)^{1/5}$

$|z^5| = \sqrt{(-243)^2 + 0^2} = +243$

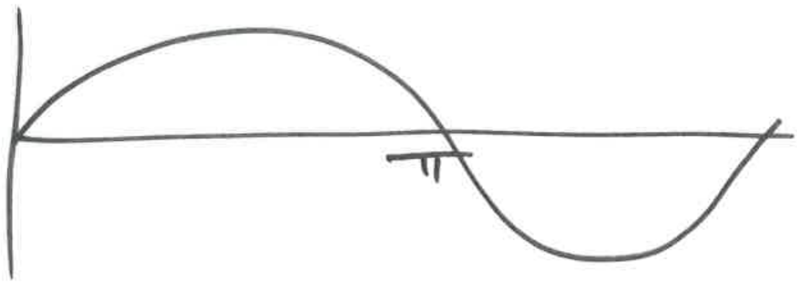
So $z^5 = 243[-1 + 0i]$

$= 243[\cos\theta + i\sin\theta]$



$\cos\theta$

$\theta = \pi$



$\theta = 0, \pi$

So $\theta = \pi$

$z^5 = 243[\cos\pi + i\sin\pi]$

or $z^5 = 243\left[\cos(\pi + 2n\pi) + i\sin(\pi + 2n\pi)\right]$

$n \in \mathbb{Z}$

$$z = 3\left[\cos\left(\frac{\pi + 2n\pi}{5}\right) + i\sin\left(\frac{\pi + 2n\pi}{5}\right)\right]$$

$$\text{or} \quad z = 3\left[\cos\left[\left(\frac{1+2n}{5}\right)\pi\right] + i\sin\left[\left(\frac{1+2n}{5}\right)\pi\right]\right]$$

with $n=0$: $z_1 = 3\left[\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right]$

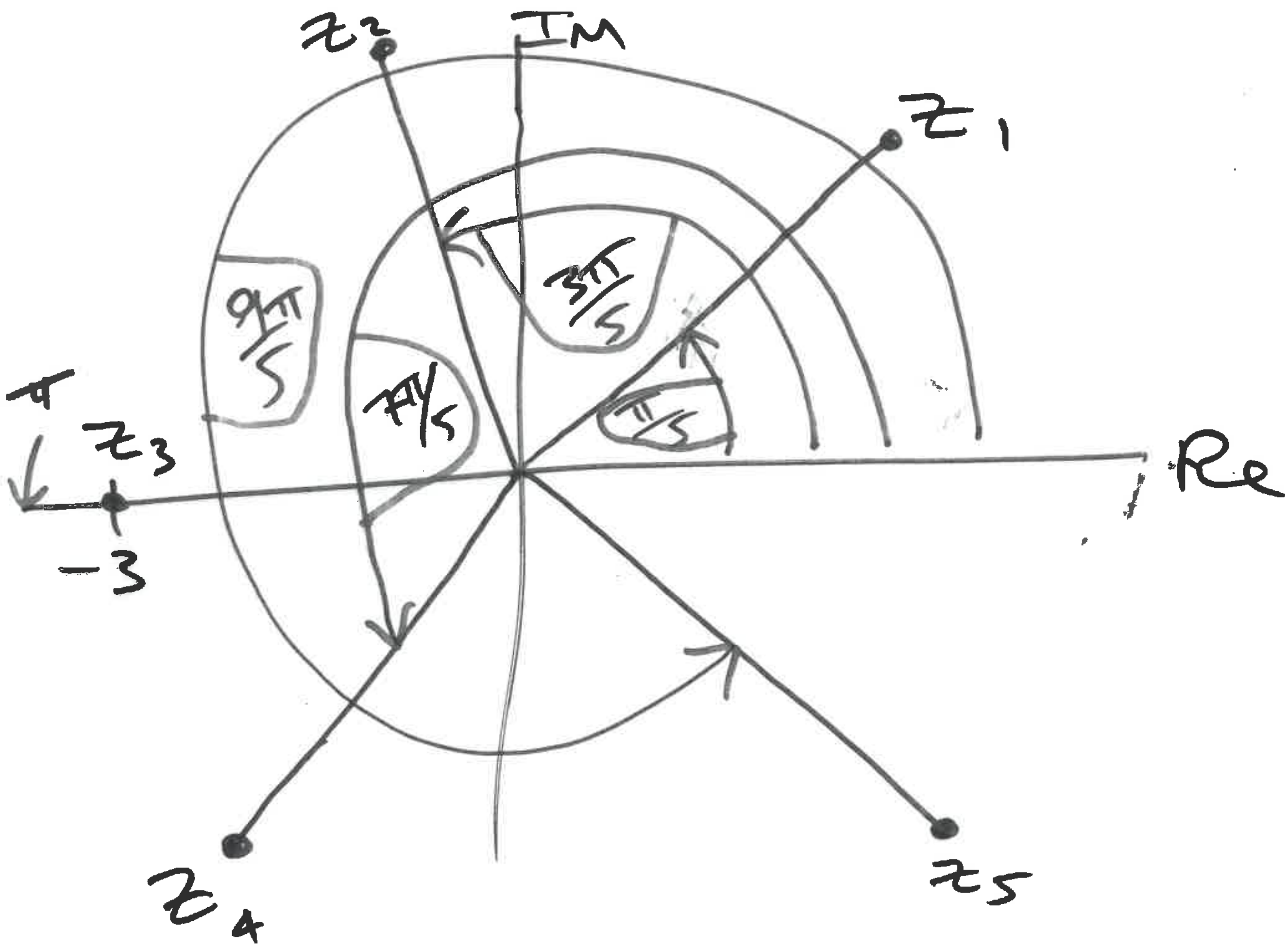
$n=1$: $z_2 = 3\left[\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right]$

$n=2$ $\quad z_3 = 3\left[\cos\pi + i\sin\pi\right] = 3(-1) = -3$

$n=3$ $\quad z_4 = 3\left[\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right]$

$n=4$ $\quad z_5 = 3\left[\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right]$

Note if you let $n=5$, you get the same result with $n=0$.



Exercise :

(a) convince yourself that all z_1, z_2, z_3, z_4, z_5 satisfy the equation (by finding powers)

(b) convince yourself using geometry (of the complex plane).