Lecture 12:

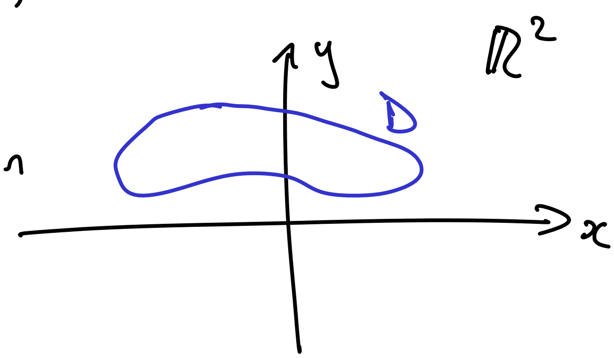
Recap: 1-variable probability distribution:

$f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$P(a \leq X \leq b) = \int_a^b f(x)\,dx$

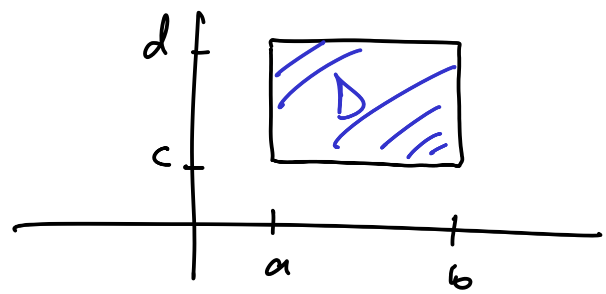
2-d case: 2 random variables X and Y

$$P((X,Y) \in D) = \iint_D f(x,y)\,dA$$



$f(x,y)$ = joint prob. distribution

In case D is a rectangle



$$P(a \leq X \leq b, c \leq Y \leq d)$$
$$= \iint_D f(x,y)\,dA$$
$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Note: For prob density we insist $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$

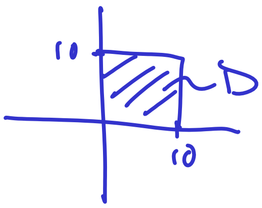
i.e. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$

Sol^n to problem posed @ end of Lect 11

Given $f(x,y) = \begin{cases} C(x+2y), & x \in [0,10],\ y \in [0,10] \\ 0, & \text{otherwise} \end{cases}$

- Since f is zero outside $[0,10] \times [0,10]$

we have $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^{10}\int_0^{10} f(x,y)\,dy\,dx$



Thus, $\displaystyle\int_0^{10}\int_0^{10} C(x+2y)\,dy\,dx = 1$ (To insist f is a prob. dist.)

Inner $\displaystyle\int_0^{10} C(x+2y)\,dy = \left. Cxy + C.\frac{2y^2}{2} \right]_0^{10}$

$= 10Cx + 100C$

Outer: $\displaystyle\int_0^{10} 10Cx + 100C\,dx = \left[10C\frac{x^2}{2} + 100Cx\right]_0^{10}$

$= 5C(100) + 100C(10)$

$= 1500C$

Thus $1500C = 1 \Rightarrow C = \frac{1}{1500}$.

(b) Compute $P(X \le 7, Y \ge 2)$

We compute $\displaystyle\int_{-\infty}^{7}\int_{2}^{\infty} f(x,y)\,dy\,dx$

$\overset{\text{(here)}}{=} \displaystyle\int_0^7\int_2^{10} f(x,y)\,dy\,dx$ here, by nature of D.

$= \displaystyle\int_0^7\int_2^{10} \frac{1}{1500}(x+2y)\,dy\,dx$

Inner: $\int_2^{10} \frac{1}{1500}(x+2y)\,dy = \frac{1}{1500}\cdot\left[xy+y^2\right]_2^{10}$

$$= \frac{1}{1500}\left(10x+100-(2x+4)\right) = \frac{1}{1500}(8x+96)$$

Outer: $\int_0^7 \frac{1}{1500}(8x+96)\,dx = \frac{1}{1500}\left[4x^2+96x\right]_0^7$

$$= \frac{1}{1500}(196+672) = \frac{868}{1500} \approx 0.579$$

$$\approx 57.9\%$$

Example (Stewart)

The average waiting time to buy a ticket for a film is 10 mins; the average waiting time to queue for popcorn is 5 mins.
Assuming these queues are independent, find the prob. that a customer waits a total of less than 20 mins altogether.

Sol^n: [Waiting time model:

$$f(t) = \begin{cases} 0 & \text{if } t<0 \\ \frac{1}{\mu}e^{-t/\mu}, & t\geq 0 \end{cases}$$

where μ = mean waiting time]

Let $X =$ waiting time for ticket purchase

$Y =$ " popcorn "

Now $f_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{10} e^{-x/10}, & x \geq 0 \end{cases}$ & $f_2(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{5} e^{-y/5}, & y \geq 0 \end{cases}$

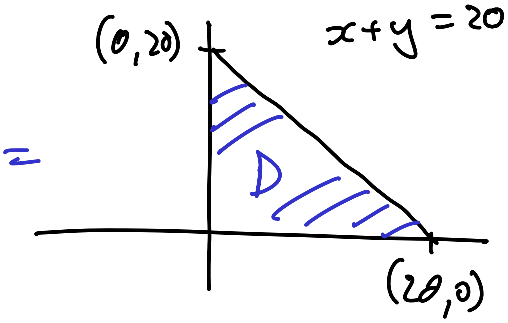
- Given that X, Y are indep.

$$\Rightarrow \quad f(x,y) = f_1(x,y)\, f_2(x,y)$$

ie $f(x,y) = \begin{cases} \frac{1}{10} e^{-x/10} \cdot \frac{1}{5} e^{-y/5} & \text{if } x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Problem: $P(X + Y < 20)$

$= P((X,Y) \in D)$ where $D =$



$$= \iint_D f(x,y)\, dA$$

= try as exercise (will append full solution below)

ie compute $\displaystyle\int_{x=0}^{20} \int_{y=0}^{20-x} f(x,y)\, dy\, dx$

$$= \int_0^{20} \int_{y=0}^{20-x} \frac{1}{50} e^{-\frac{x}{10} - \frac{y}{5}}\, dy\, dx$$

Inner: $\displaystyle\int_{y=0}^{20-x} \frac{1}{50} e^{-x/10} \cdot e^{-y/5}\, dy = \frac{e^{-x/10}}{50} \cdot \frac{e^{-y/5}}{-1/5} \Bigg]_{y=0}^{20-x}$

$$= \frac{-1}{10} e^{-x/10} \cdot \left(e^{-\frac{(20-x)}{5}} - e^{0} \right)$$

$$= \frac{-1}{10} e^{-x/10} \left(e^{-4} \cdot e^{x/5} - 1 \right)$$

Outer: $\displaystyle \frac{-1}{10} \int_0^{20} e^{-4} \cdot e^{x/10} - e^{-x/10}\, dx$ ... exercise!

Ans: ≈ 0.748

or 74.8%

Details to follow after-class (or see Stewart!!)