Ex

A moving particle starts at an initial position

$$\vec{r}(0) = \langle 1, 0, 0 \rangle$$

with initial velocity

$$\vec{v}(0) = \hat{i} - \hat{j} + \hat{k}.$$

Its acceleration is

$$\vec{a}(t) = (4t)\hat{i} + (6t)\hat{j} + k$$

find its (i) velocity and (ii) position at time t.

Solution

(i)

$$\vec{a}(t) = \vec{v}'(t)$$

So $\vec{v}(t) = \int \vec{a}(t)\, dt.$

$$\int \vec{a}(t)dt = \frac{4t^2}{2}\hat{i} + \frac{6t^2}{2}\hat{j} + t\hat{k} + \vec{c}$$

$\vec{c}$ = constant vector.

$$\therefore \vec{v}(t) = (2t^2)\hat{i} + (3t^2)\hat{j} + (t)\hat{k} + \vec{c}$$

$$\vec{v}(0) = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} + \vec{c}$$

or but $\vec{v}(0) = \hat{\imath} - \hat{\jmath} + \hat{k}$

so $\vec{c} = \hat{\imath} - \hat{\jmath} + \hat{k}$

$$\therefore \vec{v}(t) = (2t^2+1)\hat{\imath} + (3t^2-1)\hat{\jmath} + (t+1)\hat{k}$$

(ii) $\vec{r}'(t) = \vec{v}(t)$.

$$\Rightarrow \vec{r}(t) = \int \vec{v}(t)\,dt$$

$$\Rightarrow \vec{r}(t) = \int (2t^2+1)\hat{\imath} + (3t^2-1)\hat{\jmath} + (t+1)\hat{k}$$

$$\therefore \vec{r}(t) = \left(\frac{2}{3}t^3 + t\right)\hat{\imath} + (t^3 - t)\hat{\jmath} + \left(\frac{1}{2}t^2 + t\right)\hat{k} + \vec{d}$$

where $\vec{d}$ is a constant

but $\vec{r}(0) = \langle 1, 0, 0 \rangle$

so $\vec{r}(0) = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} + \vec{d}$

or $\hat{\imath} = \vec{d}$

so $\vec{r}(t) = \left(\frac{2}{3}t^3 + t + 1\right)\hat{\imath} + (t^3 - t)\hat{\jmath} + \left(\frac{1}{2}t^2 + t\right)\hat{k}$

Application 1.

Use the second law of motion

$$F = m\vec{a}$$

and
Law of Gravitation

$$F = -\frac{GMm}{r^3}\vec{r}$$

to show that:
" the orbit of a planet is in a plane "

Solution

Equating the above

$$m\vec{a} = -\frac{GMm\vec{r}}{r^3}$$

$$\Rightarrow \quad \vec{a} = -\frac{GM\vec{r}}{r^3}$$

So $\vec{a}$ is parallel to $\vec{r}$

This means

$$\vec{a} \times \vec{r} = \vec{0}$$

Recall "differentiation" rule 5

$$\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right]$$

$$= \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$$

So

$$\frac{d}{dt}\left[\vec{r}(t) \times \vec{v}(t)\right]$$

$$= \vec{r}'(t) \times \vec{v}(t) + \vec{r}(t) \times \vec{v}'(t)$$

$$= \vec{v}(t) \times \vec{v}(t) + \vec{r}(t) \times \vec{a}(t)$$

$$= \vec{0} \qquad + \qquad \vec{0}$$

$$= \vec{0}$$

Therefore: $\underline{\vec{r}(t) \times \vec{v}(t) = \vec{h}}$

where $\vec{h}$ is a constant vector.

We are assuming $\vec{r}(t) \times \vec{v}(t) \neq \vec{0}$

i.e. $\vec{r}(t)$ and $\vec{v}(t)$ are not parallel

So $\vec{r}(t) \times \vec{v}(t) = \vec{h}$

Means that $\vec{r}(t)$ is perpendicular to the <u>constant</u> vector $\vec{h}$

i.e. $\vec{r}(t)$ is in a plane with normal $\vec{h}$ (constant)

Functions of two variables

Definition:

A function of two variables

is a rule

that assigns to each

ordered pair of real numbers

(x, y) in a set D

a unique real number

denoted by $f(x, y)$.

The set D is the domain

of f

and the range of f is

the set of values that

f takes on

that is $\left\{ f(x, y) \mid (x, y) \in D \right\}$.

Note :

- We write $z = f(x, y)$
to make explicit the value
taken on by f
at the general point (x, y).

- The variables x and y
are independent variables
and z is the dependent variable

Ex

Find the domain of

$$f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$$

Solution

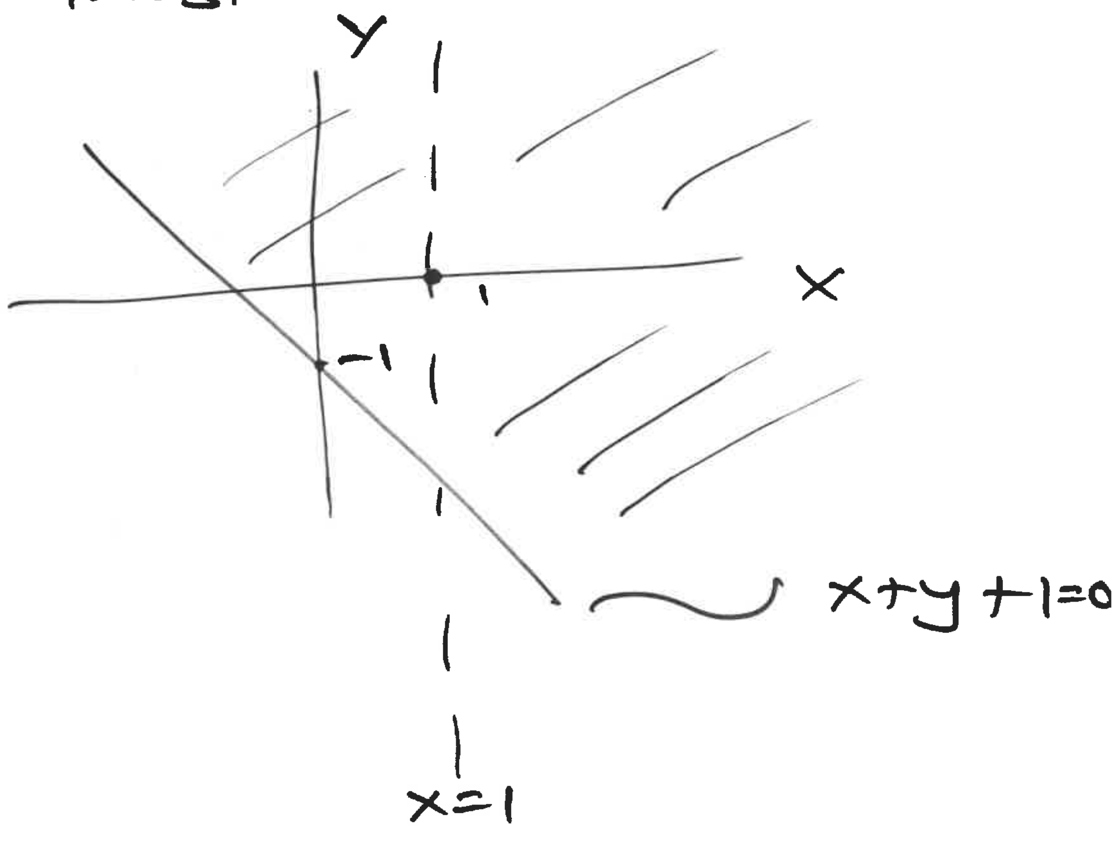
$$D = \{ (x, y) \mid x+y+1 \geq 0, x \neq 1 \}$$

The inequality $x+y+1 \geq 0$
or $y \geq -x-1$

describes the points on or above the line:

$$y = x - 1,$$

while $x \neq 1$ means that the points on the line $x = 1$ must be excluded.



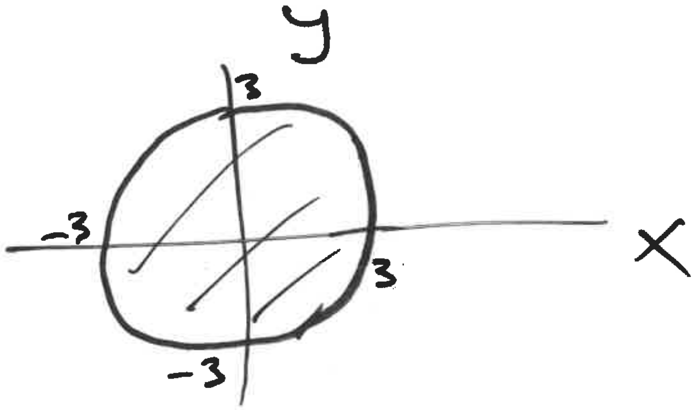
<u>Ex</u>

Find the domain and range of

$$g(x, y) = \sqrt{9 - x^2 - y^2}$$

Solution

$$D = \{ (x, y) \mid 9 - x^2 - y^2 \geq 0 \}$$
$$= \{ (x, y) \mid x^2 + y^2 \leq 9 \}$$



Since z is a positive square root, $z \geq 0$

The largest value of $z(x,y)$ is $z(0,0) = +3$

So $0 \leq z \leq 3$.

$\therefore$ Range $= \{ z \mid 0 \leq z \leq 3 \}$

Ex

Sketch the graph of the function:

$f(x, y) = 6 - 3x - 2y$

Solution

The graph of f has the equation

$$z = 6 - 3x - 2y$$

which represents a plane!

$$3x + 2y + z = 6$$

(To graph it find some intercepts with the main axis)

Ex

Sketch the graph of

$$g(x,y) = \sqrt{9 - x^2 - y^2}$$

Solution

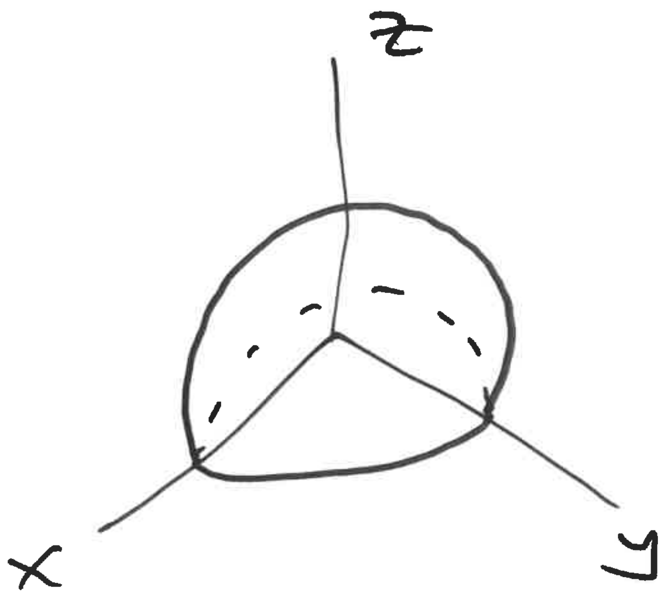
$$z = \sqrt{9 - x^2 - y^2}$$

$$\Rightarrow \quad z^2 = 9 - x^2 - y^2$$

or $\quad x^2 + y^2 + z^2 = 9$ (sphere)

As $z \geq 0$

the graph of $g(x,y)$ is the top half of this sphere



Definition

The level curves of a function f of two variables are the curves with equations

$$f(x,y) = k$$

where k is a constant.

Ex

Sketch the level curves of the function

$$g(x,y) = \sqrt{9-x^2-y^2}$$

for $z = 0, 1, 2, 3$.

Solution

$z = 0$

$$0 = \sqrt{9 - x^2 - y^2}$$

or $\quad x^2 + y^2 = 9 \rightarrow$ Circle of radius 3 centre $(0,0)$

$z = 1$

$$1 = \sqrt{9 - x^2 - y^2}$$
$$1 = 9 - x^2 - y^2$$
$$x^2 + y^2 = 8 = (2\sqrt{2})^2$$

$\rightarrow$ Circle of radius $2\sqrt{2}$ and centre $(0,0)$

$z = 2$

$$2 = \sqrt{9 - x^2 - y^2}$$
$$4 = 9 - x^2 - y^2$$
$$x^2 + y^2 = 5 \rightarrow$$ Circle of radius $\sqrt{5}$ + centre $(0,0)$

$z = 3$

$$3 = \sqrt{9 - x^2 - y^2}$$
$$x^2 + y^2 = 0 \rightarrow$$ point $(0,0)$