Definition: <u>The arc length</u>

If $\vec{r}(t) = f(t)\,\hat{\imath} + g(t)\,\hat{\jmath} + h(t)\,\hat{k}$

then the arc length from

$\vec{r}(a)$ to $\vec{r}(b)$ is:

$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

Provided $f'(t)$, $g'(t)$ and $h'(t)$ are continuous.

<u>Ex</u> Find the length of the arc of

$\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\,\hat{k}$

from the point $(1, 0, 0)$

to the point $(1, 0, 2\pi)$

<u>Sol</u>

$\vec{r}\,'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + 1\,\hat{k}$

$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}$

$= \sqrt{(\sin^2 t + \cos^2 t) + 1}$

$= \sqrt{1 + 1} = \sqrt{2}$

Then the arc from $(1, 0, 0)$ to $(1, 0, 2R)$ is described by the parameter interval

$$0 \leq t \leq 2R$$

and so we have

$$L = \int_0^{2R} |\vec{r}\,'(t)|\, dt$$

$$= \int_0^{2R} \sqrt{2}\, dt$$

$$= \sqrt{2}\, t \Big|_0^{2R} = \sqrt{2}\,(2R)$$

Definition:

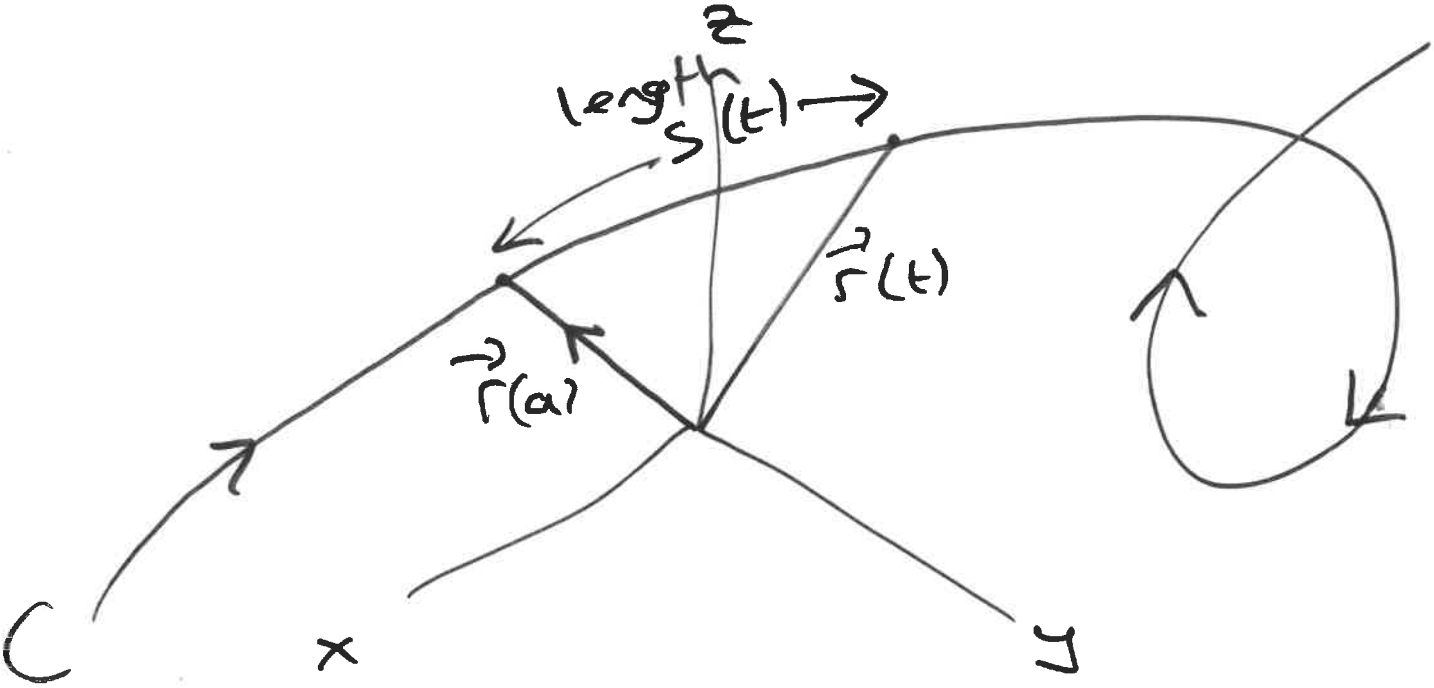
Assume that C is a curve given by a vector function

$$\vec{r}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$$
$$a \leq t \leq b$$

where $\vec{r}\,'(t)$ is continuous and C is traversed once as t goes from a to b, its arc <u>length s</u> is defined by

$$s(t) = \int_a^t |\vec{r}\,'(u)|\, du \, .$$

Hence $s(t)$ is the length of the part of C between $\vec{r}(a)$ and $\vec{r}(t)$.



A single curve can be parameterised by more than one vector function.

For example, the helix:

(i) $\vec{r}_1(t) = \cos t \ \hat{\imath} + \sin t \ \hat{\jmath} + t \ \hat{k}$

$R \leq t \leq 2R$

could also be represented by the function:

(ii) $\vec{r}_2(u) = \cos(e^u)\hat{\imath} + \sin(e^u)\hat{\jmath} + e^u \hat{k}$

$\ln(R) \leq u < \ln(2R)$

where the connection between the parameters t and u is given by $t = e^u$.

Equations (i) and (ii) are parameterizations of the curve C.

Note that

$$L = \int_R^{2R} |\vec{r}_1{}'(t)| dt = \int_{\ln(R)}^{\ln(2R)} |\vec{r}_2{}'(u)| du.$$

If we use

$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

to compute arc length the answer is independent of the parametrization that is used.

Note:

- it is often useful to parametrize a curve with respect to arc length;

  because arc length arises naturally from the shape of the curve;

  and does not depend on a particular coordinate system.

Ex

Reparametrization of the helix

$$\vec{r}(t) = (\cos t)\,\hat{i} + (\sin t)\,\hat{j} + (t)\,\hat{k}$$

with respect to arc length measured from $(1,0,0)$ in the direction of increasing t.

Solution

The initial point $(1,0,0)$ corresponds to the parameter value $t=0$,

$$s = s(t) = \int_0^t \underbrace{|\vec{r}\,'(u)|}_{\text{saw before } = \sqrt{2}}\,du$$

$$s = s(t) = \int_0^t \sqrt{2}\,du = \sqrt{2}\,u\Big|_0^t$$

$$s = s(t) = \sqrt{2}\,t$$

Therefore, $t = \dfrac{s}{\sqrt{2}}$

and the required reparametrization is obtained by substituting for t:

$$\vec{r}(s) = \cos\left(\frac{s}{\sqrt{2}}\right)\hat{\imath} + \sin\left(\frac{s}{\sqrt{2}}\right)\hat{\jmath} + \frac{s}{\sqrt{2}}\hat{k}.$$

Definition

The <u>curvature</u> of C at a given point is a measure of how quickly the curve C changes direction at that point.

Its value is

$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

Ex Show that the curvature of a circle of radius a is $\frac{1}{a}$.

Solution

The position vector for a circle

$$\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$$

$$\vec{r}\,'(t) = -a\sin t\,\hat{i} + a\cos t\,\hat{j}$$

$$|\vec{r}\,'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2}$$

$$= a \quad *$$

$$\vec{T}(t) = \frac{r'(t)}{|r'(t)|}$$

$$= \frac{-a\sin t\,\hat{i} + a\cos t\,\hat{j}}{\sqrt{(-a\sin t)^2 + (a\cos t)^2}}$$

$$= \frac{-a\sin t\,\hat{i} + a\cos t\,\hat{j}}{a}$$

$$\therefore \vec{T}(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

$$\Rightarrow \vec{T}\,'(t) = -\cos t\,\hat{i} - \sin t\,\hat{j}$$

$$|\vec{T}\,'(t)| = 1 \quad **$$

$$K = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{1}{a}$$

Motion in space: velocity and acceleration

Definition

If the position vector for a moving object is given by $\vec{r}(t)$, then the velocity vector $\vec{v}(t)$ is given by $\vec{r}'(t)$.

The speed of the moving object is: $|\vec{v}(t)|$

$= |r'(t)|$

The acceleration of the moving object is $\vec{a}(t)$, where

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$$

Application of vector functions

<u>Ex</u>

The position vector of an object moving is given by

$$\vec{r}(t) = (t^4 - 1)\hat{i} + t^3\hat{j} + 0\hat{k}$$

Find its
(i) velocity
(ii) speed
(iii) acceleration
when $t = 1$.

(i) Velocity $= \vec{r}\,'(t)$

$$\vec{r}\,'(t) = (4t^3)\hat{i} + (3t^2)\hat{j} + 0\hat{k}$$

$$\vec{r}\,'(1) = \left[4.(1)^3\right]\hat{i} + \left[3.(1)^2\right]\hat{j} + 0\hat{k}$$

$$= 4\hat{i} + 3\hat{j} + 0\hat{k}$$

(ii) Speed $= |\vec{r}(t)|$

$$\vec{r}(1) = 4\hat{i} + 3\hat{j} + 0\hat{k}$$

So speed at $t=1$ is:

$$|4\hat{i} + 3\hat{j} + 0\hat{k}|$$
$$= \sqrt{4^2 + 3^2 + 0^2}$$
$$= \sqrt{16+9} = \sqrt{25} = 5.$$

(iii) $\vec{a}(t) = \vec{v}\,'(t)$

but $\vec{v}(t) = (4t^3)\hat{i} + (3t^2)\hat{j} + (0)\hat{k}$

$$\vec{a}(t) = (12t^2)\hat{i} + (6t)\hat{j} + 0\hat{k}$$
$$\vec{a}(1) = [12.(1)^2]\hat{i} + [(6)(1)]\hat{j} + [0]\hat{k}$$
$$\vec{a}(1) = 12\hat{i} + 6\hat{j} + 0\hat{k}$$