Definition

Unit vectors along the axis are vectors of length 1

- in the x-axis direction which is called: $\hat{i}$
- in the y-axis direction which is called: $\hat{j}$
- in the z-axis direction which is called: $\hat{k}$

Example

we can write the vector

$$\langle 4, 5, -9 \rangle$$

as $4\hat{i} + 5\hat{j} - 9\hat{k}$

The unit vector in the direction of $\vec{v}$ is:

$$\frac{\vec{v}}{|\vec{v}|}$$

Example

Find the unit vector in the direction of

$\vec{v} = 2\hat{i} - \hat{j} - 2\hat{k}$

Solution

$$|\vec{v}| = \sqrt{2^2 + (-1)^2 + (-2)^2}$$
$$= \sqrt{4 + 1 + 4}$$
$$= \sqrt{9}$$
$$= 3$$

So $\dfrac{\vec{v}}{|\vec{v}|} = \dfrac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$

Sometimes $\dfrac{\vec{v}}{|\vec{v}|}$ is written as $\hat{v}$

Definition

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$

and $\vec{b} = \langle b_1, b_2, b_3 \rangle$

then the dot product of

$\vec{a}$ and $\vec{b}$

is given by the number

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example

Find $\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle$

$$= (-1)(6) + (7)(2) + 4\left(-\tfrac{1}{2}\right)$$

$$= -6 + 14 - 2$$

$$= 6$$

Exercise

Find

1) $\langle 3, 7, -2 \rangle \cdot \langle 4, 0, 0 \rangle$

2) $\hat{i} + \hat{j} + \hat{k} \cdot \hat{i} - \hat{j}$

3) $7\hat{i} + 3\hat{j} - \hat{k} \cdot \hat{k}$

Properties of the dot product

1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(d\vec{a}) \cdot \vec{b} = d(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (d\vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$

Exercise

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{c} = \langle c_1, c_2, c_3 \rangle$

and $d \in \mathbb{R}$

Show the 5 properties of the dot product

i.e. find expressions for the LHS and RHS then show LHS = RHS.

Theorem:

If θ is the angle between vectors $\vec{a}$ and $\vec{b}$,

then

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta .$$

So $$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Example Find the angle between the vectors

$\vec{a} = \langle 2, 2, -1 \rangle$ and $\vec{b} = \langle 5, -3, 2 \rangle$

Solution

$$\vec{a} \cdot \vec{b} = \langle 2, 2, -1 \rangle \cdot \langle 5, -3, 2 \rangle$$

$$= (2)(5) + (2)(-3) + (-1)(2)$$

$$= 10 - 6 - 2$$

$$= 2$$

$$|\vec{a}| = \sqrt{2^2 + 2^2 + (-1)^2}$$

$$= \sqrt{4 + 4 + 1}$$

$$= \sqrt{9}$$

$$= 3$$

$$|\vec{b}| = \sqrt{5^2+(-3)^2+2^2}$$
$$= \sqrt{25+9+4}$$
$$= \sqrt{38}$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{2}{3\sqrt{38}}$$

$$\theta = \cos^{-1}\frac{2}{3\sqrt{38}} \doteqdot 83.8^\circ$$

Exercise

Find the angle between the vectors

$$\langle 4, -3, 0\rangle \text{ and } \langle -5, -12, 0\rangle$$

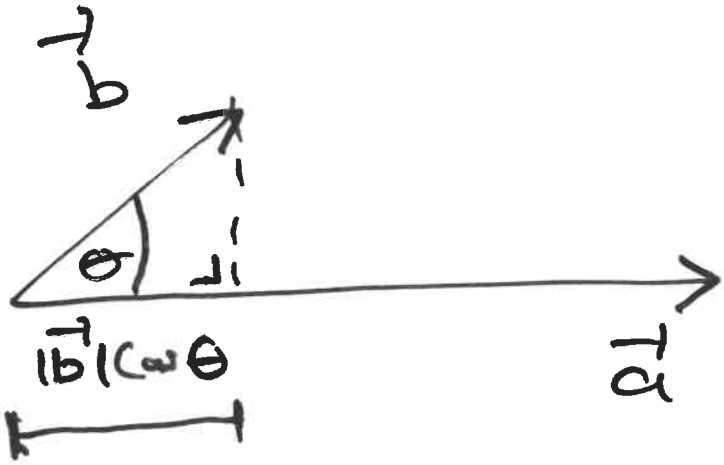
Exercise

Show that the vectors

$$2\hat{i} + 2\hat{j} - \hat{k} \text{ and}$$
$$5\hat{i} - 4\hat{j} + 2\hat{k}$$

are perpendicular

The projection of $\vec{b}$ on to $\vec{a}$.



$|\vec{b}|\cos\theta$ is the scalar projection of $\vec{b}$ on to $\vec{a}$.

(also called the component of $\vec{b}$ along $\vec{a}$).

Recall

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta.$$

So $$|\vec{b}|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \mathrm{Comp}_{\vec{a}}\vec{b}$$

The vector projection of $\vec{b}$ on to $\vec{a}$

$$\mathrm{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\frac{\vec{a}}{|\vec{a}|}$$

or $\quad \text{proj}_{\vec{a}} \vec{b} = \left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a}$

Example

Find the vector projection of $\vec{b} = \langle 1, 1, 2 \rangle$ on to $\vec{a} = \langle -2, 3, 1 \rangle$

Solution

$$\vec{a} \cdot \vec{b} = (1)(-2) + (1)(3) + (2)(1)$$
$$= -2 + 3 + 2 = 3$$

$$|\vec{a}| = \sqrt{(-2)^2 + 3^2 + 1^2}$$
$$= \sqrt{4 + 9 + 1}$$
$$= \sqrt{14}$$

So $\text{proj}_{\vec{a}} \vec{b} = \dfrac{3}{14} \langle -2, 3, 1 \rangle$

Definition

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$

and $\vec{b} = \langle b_1, b_2, b_3 \rangle$

then the cross product of $\vec{a}$ and $\vec{b}$

$$\vec{a} \times \vec{b} =$$

$$\langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

Alternatively

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Properties of the cross product:

1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

2) $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$

3) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

4) $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c} + \vec{b}\times\vec{c}$

5) $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

6) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$

<u>Definition</u>

The product $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called the <u>scalar triple product</u> of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

if $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

and $\vec{c} = \langle c_1, c_2, c_3 \rangle$

then $\vec{a}\cdot(\vec{b}\times\vec{c})$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Exercise 1

if

$$\vec{a} = \langle 2, 1, -1 \rangle$$
$$\vec{b} = \langle 1, 3 \; -7 \rangle$$
$$\vec{c} = \langle 0, 1 \; 4 \rangle$$

Find $\vec{a} \cdot (\vec{b} \times \vec{c})$

Exercise 2

if

$$\vec{a} = \langle 7, -1, 4 \rangle$$
$$\vec{b} = \langle -3, 2, -7 \rangle$$
$$\vec{c} = \langle 4, 1, -3 \rangle$$

Find $\vec{a} \cdot (\vec{b} \times \vec{c})$

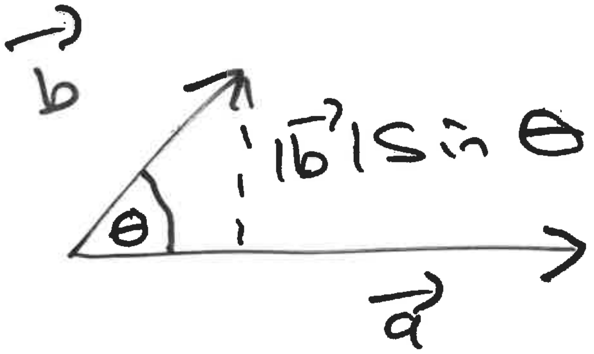
$\vec{a} \times (\vec{b} \times \vec{c})$ has a geometrical meaning

$\rightarrow$ see later.

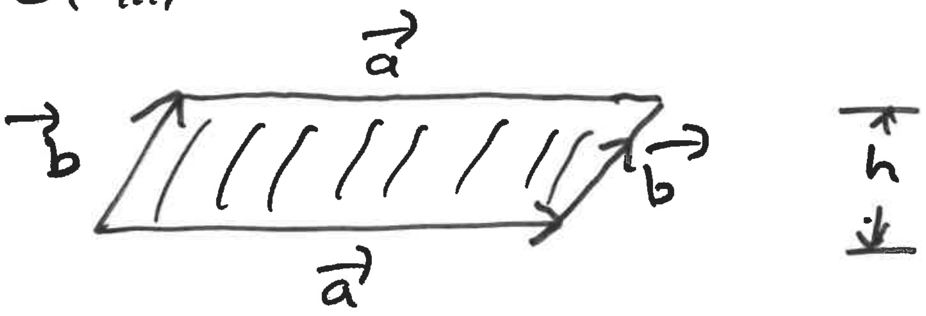
Theorem

If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta.$$



Area of ////



$$\begin{aligned}\text{Area of ////} &= |\vec{a}|\, h \\ &= |\vec{a}| \left[|\vec{b}| \sin\theta \right] \\ &= \vec{a} \times \vec{b}\end{aligned}$$

Note:

$\vec{a} \times \vec{b}$ is orthogonal / perpendicular to both $\vec{a}$ and $\vec{b}$

<u>Exercise</u>

if $\vec{a} = \langle a_1, a_2, a_3 \rangle$

and $\vec{b} = \langle b_1, b_2, b_3 \rangle$

<u>Find</u> $\vec{a} \times \vec{b}$

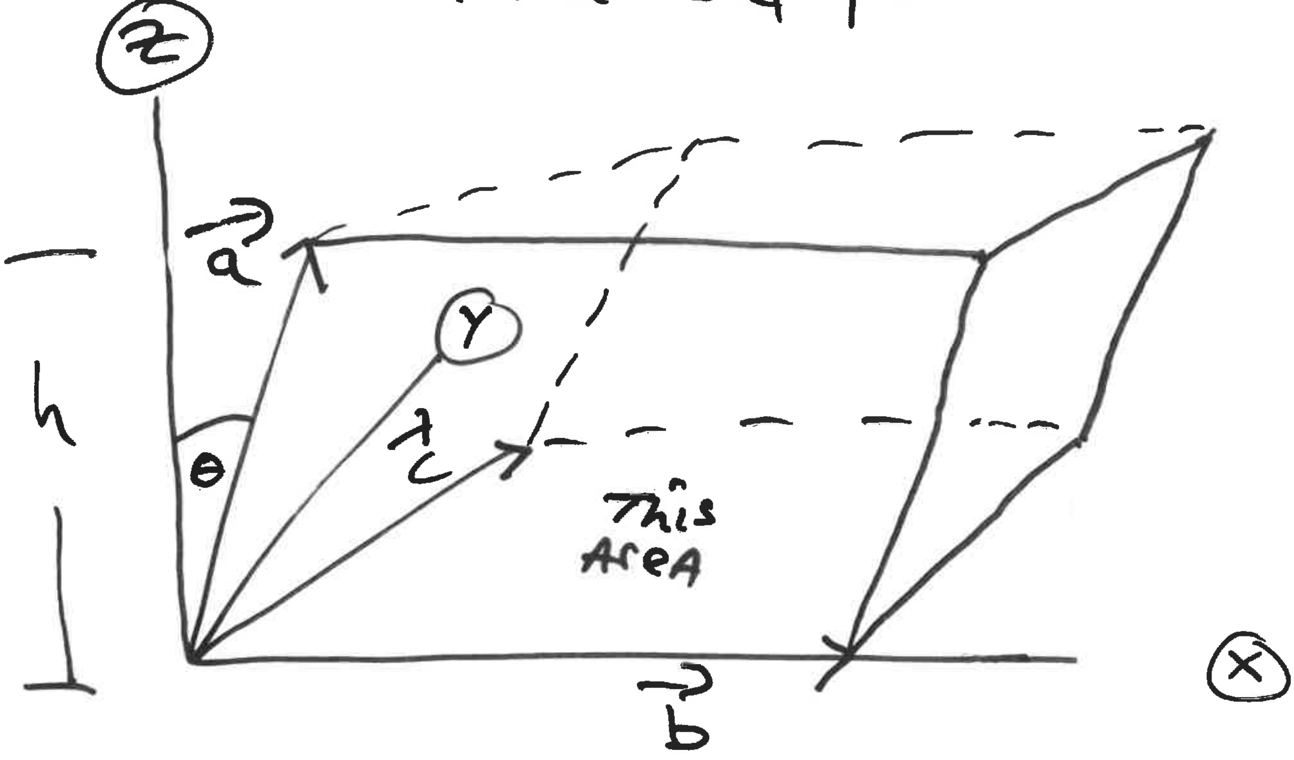
<u>Show</u> $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$

i.e. $\vec{a}$ and $\vec{a} \times \vec{b}$ are perpendicular

Show $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$

i.e. $\vec{b}$ and $\vec{a} \times \vec{b}$ are perpendicular

Parallelepiped



"deformed cuboid"

Volume of the parallelepiped

$= (h)($"THIS AREA"$)$

$h = |\vec{a}| \cos[$Angle between $\vec{a}$ and z-axis$)$

THIS AREA $= |\vec{b} \times \vec{c}|$

$\vec{b}$ and $\vec{c}$ are in the x-y plane

$\Rightarrow$ $\vec{b} \times \vec{c}$ lie along the z-axis

So $h = |\vec{a}| \cos[$Angle between $\vec{a}$ and $\vec{b}\times\vec{c}]$

Volume

$= |\vec{a}| \cos[$Angle between $\vec{a}$ and $\vec{b}\times\vec{c}]\ |\vec{b}\times\vec{c}|$

$= \vec{a}\cdot(\vec{b}\times\vec{c})$

Note:

If the volume $=0$, then the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ lie in the same plane

Example

Show that the vectors $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$ and $\vec{c} = \langle 0, -9, 18 \rangle$ are coplanar.

Solution

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$$= (-1)(18) - (-9)(4)$$
$$-4\left[(2)(18) - (0)(4)\right]$$
$$-7\left[(2)(-9) - (-1)(0)\right]$$

$$= \quad -18 + 36$$
$$-4\,[36]$$
$$-7\,[-18]$$

$$= \quad 18$$
$$-144$$
$$+126$$
$$= 0$$

$\therefore$ Volume of parallelepiped $= 0$

So $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar

Example

Find a unit vector perpendicular to the plane containing:

$P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.

Solution

$\overrightarrow{PQ}$ lies along the plane

$\overrightarrow{PQ} = \vec{Q} - \vec{P} = (2-1,\ 1-(-1),\ -1-0)$

$$= (1, 1+1, -1)$$

$$\therefore \overrightarrow{PQ} = (1, 2, 1)$$

$\overrightarrow{QR}$ lies along the plane

$$\overrightarrow{QR} = \vec{R} - \vec{Q} = (-1-2, 1-1, 2-(-1))$$

$$= (-3, 0, 2+1)$$

$$\overrightarrow{QR} = (-3, 0, 3)$$

$\overrightarrow{PQ} \times \overrightarrow{QR}$ is perpendicular to the plane

(as it is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ which lie in the plane)

$$\overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ -3 & 0 & 3 \end{vmatrix}$$

$$= \hat{i}\left[(2)(3) - (0)(1)\right] - \hat{j}\left[(1)(3) - (-3)(1)\right]$$
$$+ \hat{k}\left[(1)(0) - (-3)(2)\right]$$

$$\overrightarrow{PQ} \times \overrightarrow{QR} = \hat{i}[6] - \hat{j}[3+3] + \hat{k}[6]$$

$$= \langle 6, -6, 6 \rangle$$

$$\|\overrightarrow{PQ} \times \overrightarrow{QR}\| = \sqrt{(6)^2 + (-6)^2 + 6^2}$$

$$= \sqrt{36 + 36 + 36}$$

$$= \sqrt{(36)(3)}$$

$$= \sqrt{36}\sqrt{3} = 6\sqrt{3}$$

$$\frac{\overrightarrow{PQ} \times \overrightarrow{QR}}{\|\overrightarrow{PQ} \times \overrightarrow{QR}\|} = \frac{1}{6\sqrt{3}} \langle 6, -6, 6 \rangle$$

$$= \frac{1}{\sqrt{3}} \langle 1, -1, 1 \rangle$$

Exercise

In the example above show that

$\overrightarrow{PQ} \times \overrightarrow{QR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.