MA211 – Calculus I

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MA211 – assessment

50% 2hr main university exam

30% 3 in-class tests
(1st on Monday week 4)

20% communication skills test

100%

Book : "Calculus" by
James Stewart.

- Notes will be placed on
Blackboard
- Course web site
www.maths.nuigalway.ie/~mstudies/ma211
- Tutorials start : week 3

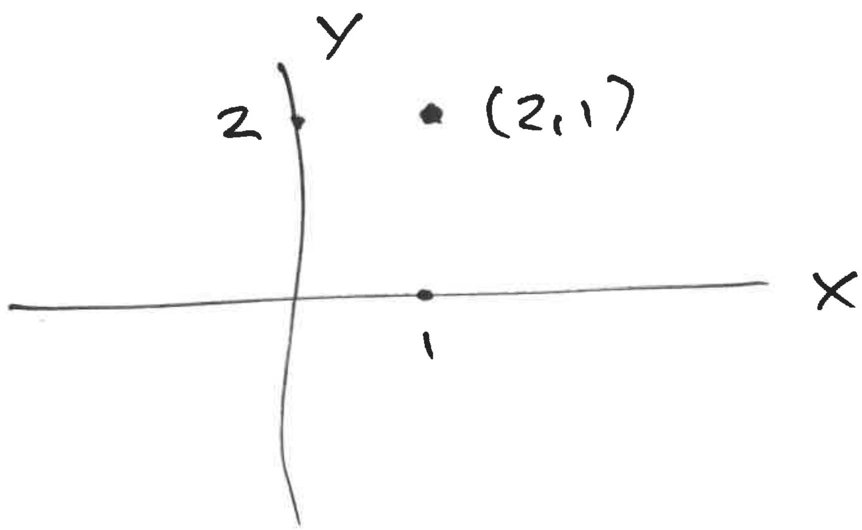
The Course Outline

- Vectors
- Partial derivatives

Vectors

We need two numbers to represent a point in a plane.

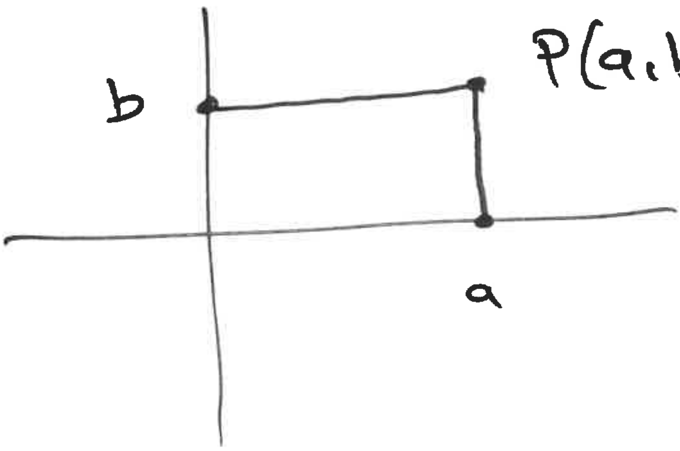
For example



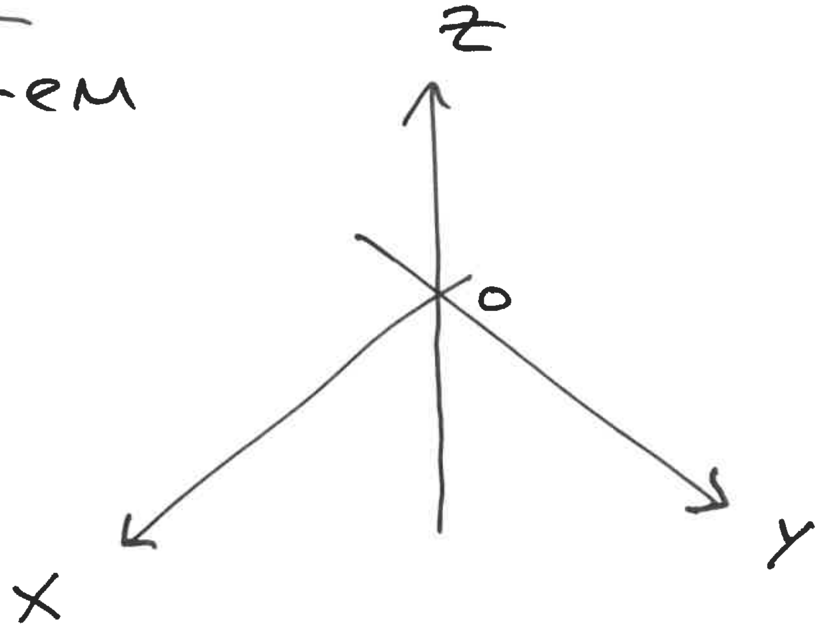
In general to locate a point P in the plane

we use an ordered pair (a,b) of real numbers, with

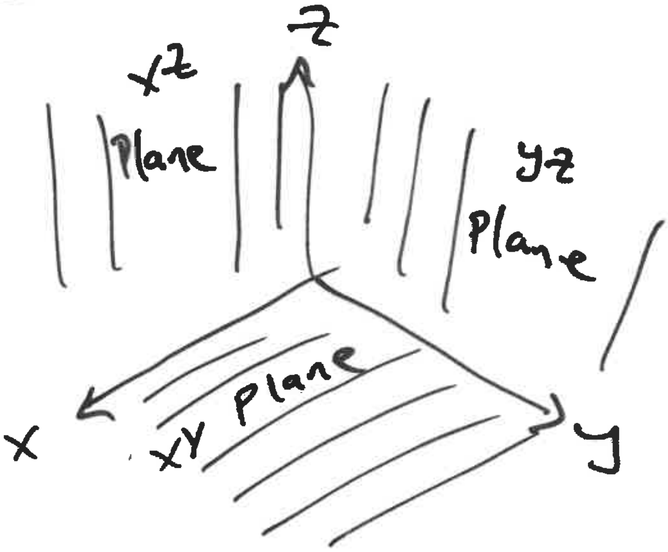
- a is the x-coordinate
- b is the y-coordinate



Three - dimensional coordinate system



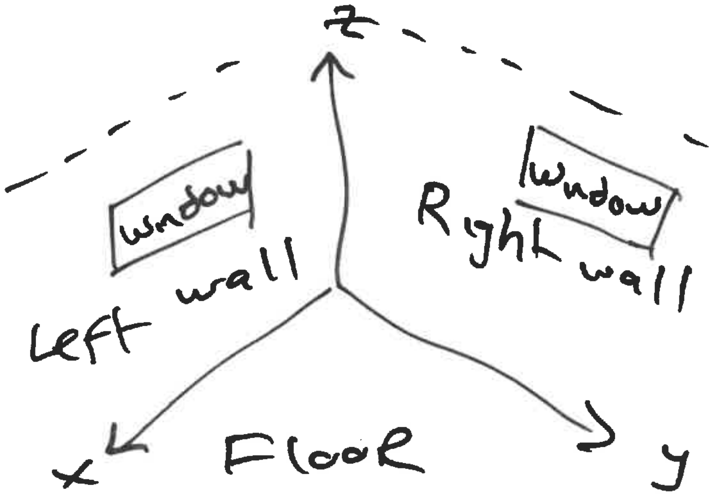
Coordinate Planes



The three coordinate planes divide space into eight parts called octants.

Coordinate Planes

e.g room corner

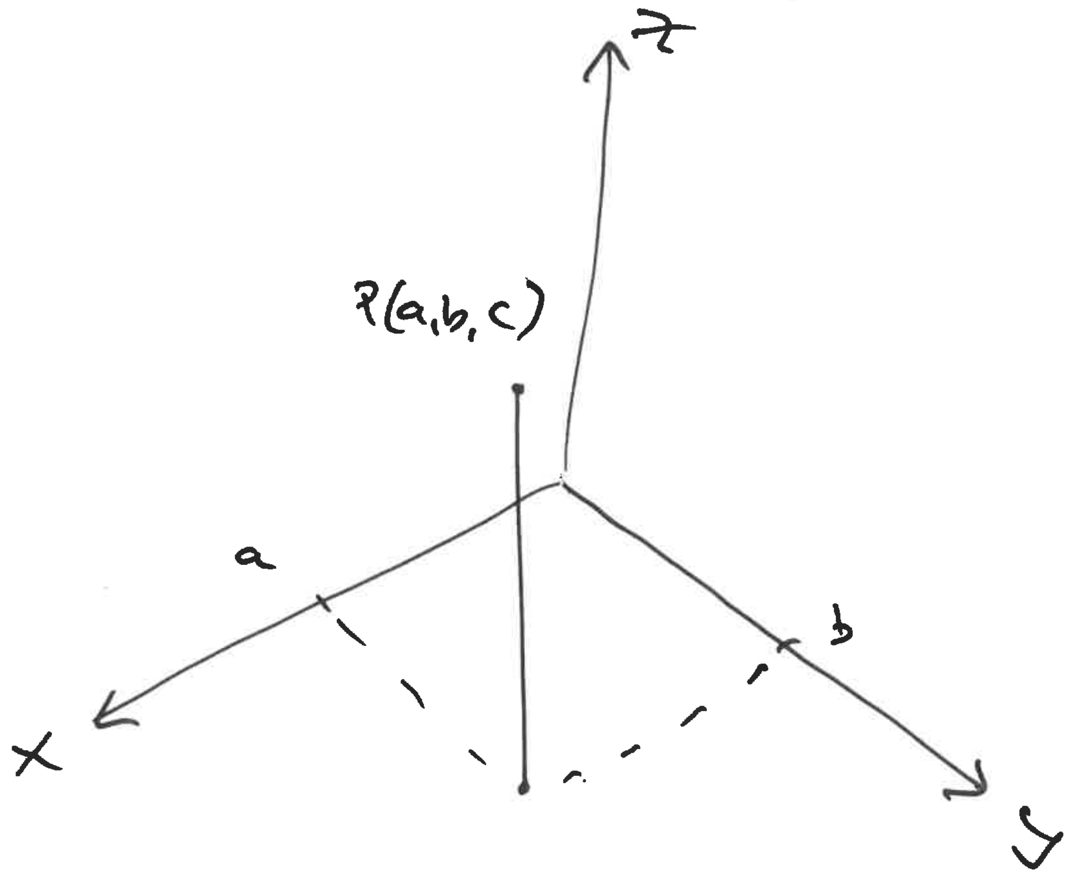


Coordinates

The point P represented by the ordered triple (a, b, c)

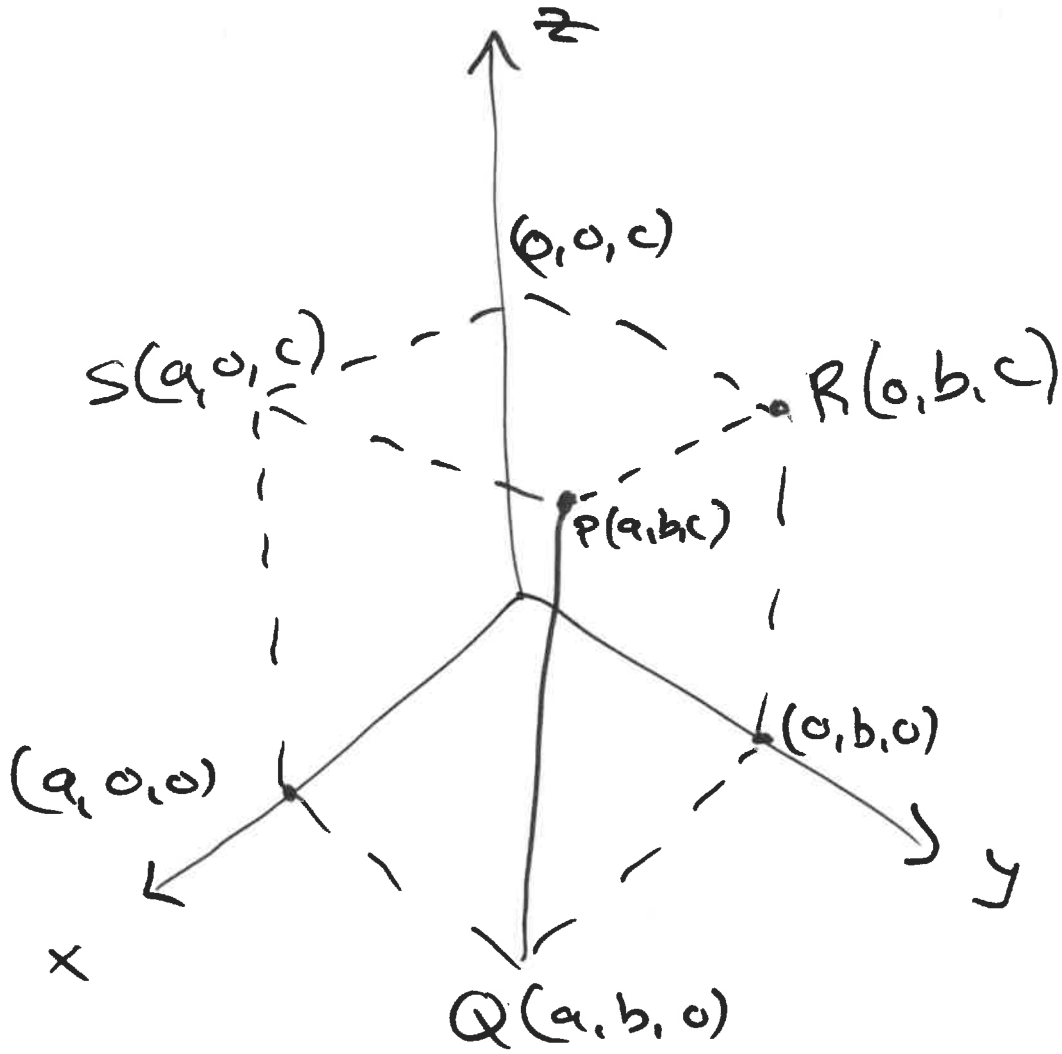
a, b, c are coordinates of P where

- a is the x – coordinate
- b is the y – coordinate
- c is the z – coordinate



The point P is represented by the ordered triple (a, b, c)

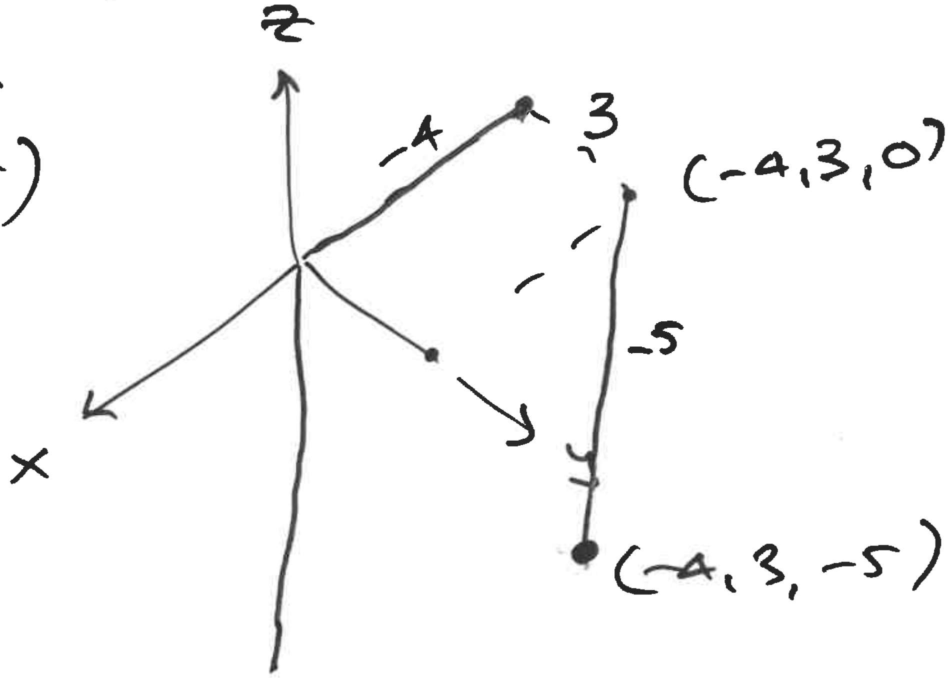
- Q is the projection of P on the xy-plane



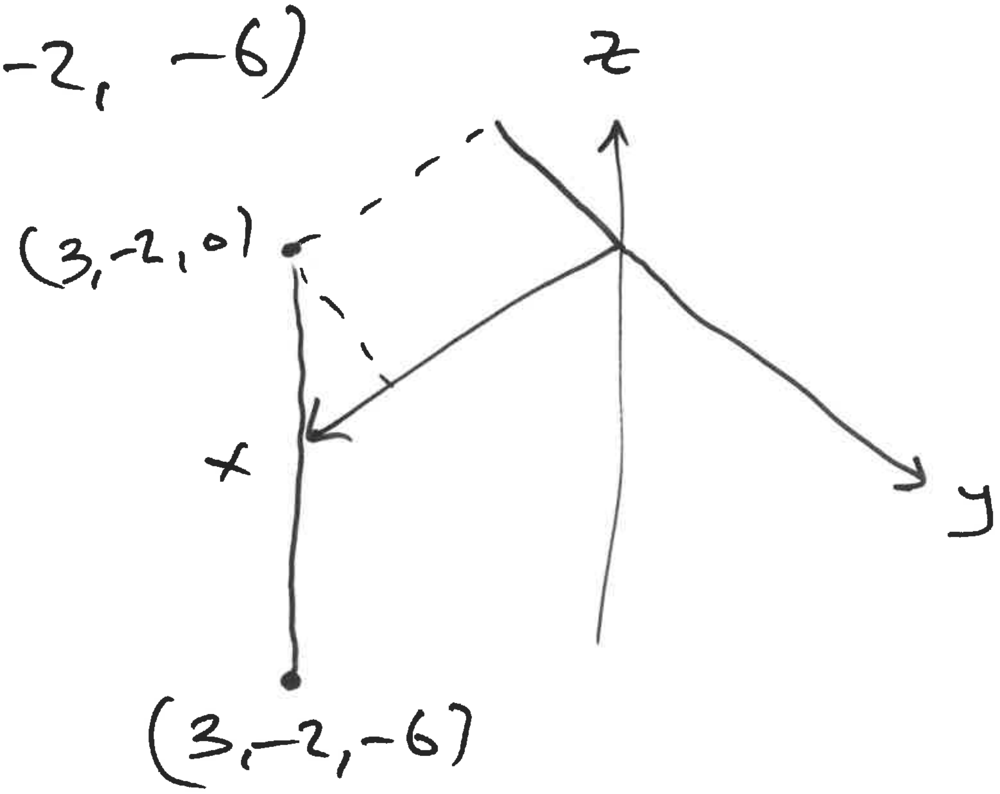
- R is the projection of P on the yz-plane
- S is the projection of P on the xz-plane.

Examples

The point $(-4, 3, -5)$



The point $(3\ -2, -6)$



Exercises

Exercise 1.

Suppose you start at the origin,
move along the x-axis
4 units in the positive direction
and then move downwards a distance of 3 units.

What are the coordinates of your position?

Exercise 2.

Sketch the points
$(0,5,2)$, $(4,0,-1)$,
$(2,4,6)$ and $(1,-1,2)$

on a single set of coordinate axes.

Three-dimensional rectangular coordinate system

- $\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

is the set of all ordered triples of real numbers and is denoted $\mathbb{R}^3$ [pronounced: "R three"]

- We have established a one-to-one correspondence between points P in space and ordered triples (a, b, c) in $\mathbb{R}^3$.

Its called a three-dimentional rectangular coordinate system.

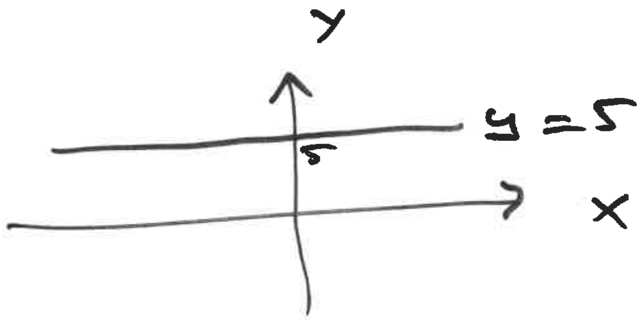
Example 1 : Sketch

1. $y = 5$ in $\mathbb{R}^2$

2. $y = 5$ in $\mathbb{R}^3$

Solution

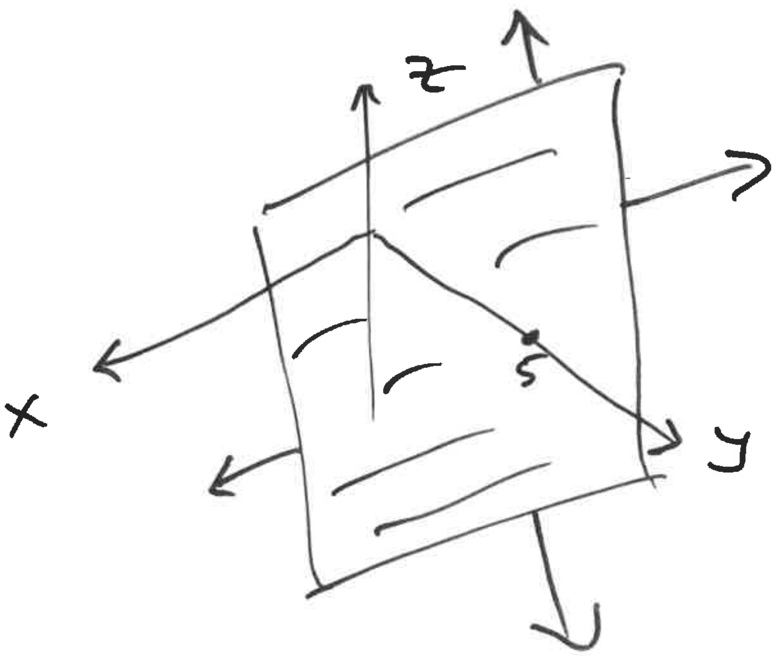
1.



$\therefore$ Line

i.e. a Curve

2.

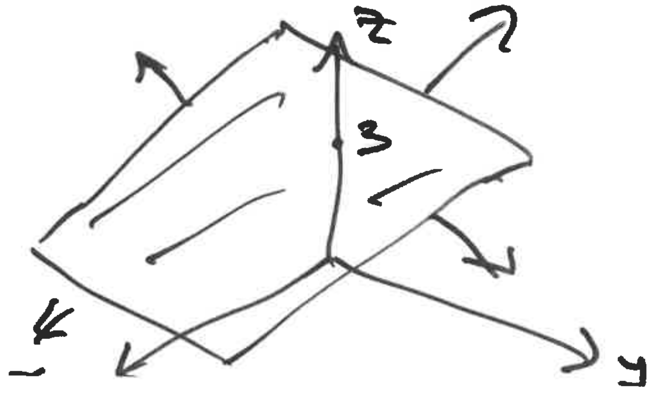


$\therefore$ Plane

i.e. a Surface

Example 2 : Sketch

$z = 3$ in $\mathbb{R}^3$



Exercise : Sketch

1. $x = 6$ in $\mathbb{R}^2$
2. $x = 6$ in $\mathbb{R}^3$.

SUMMARY

In two dimensions, the graph of an equation involving x and y is a <u>curve</u> in $\mathbb{R}^2$

In three dimensions, an equation in x, y and z represents <u>a</u> surface in $\mathbb{R}^3$.

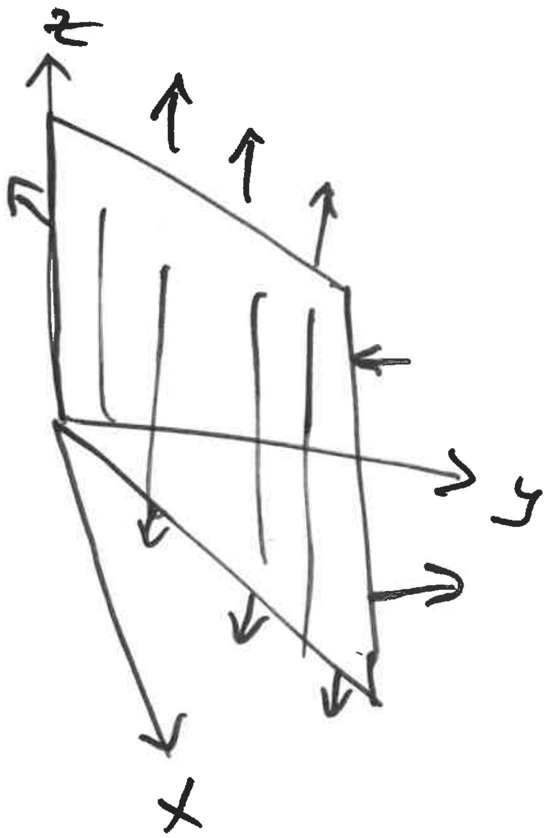
If k is a constant

- $x = k$ in $\mathbb{R}^3$ is a plane parallel to the yz-plane
- $y = k$ in $\mathbb{R}^3$ is a plane parallel to the xz-plane
- $z = k$ in $\mathbb{R}^3$ is a plane parallel to

Example:

What about the surface in $\mathbb{R}^3$ represented by the equation

$$y = x \; ?$$



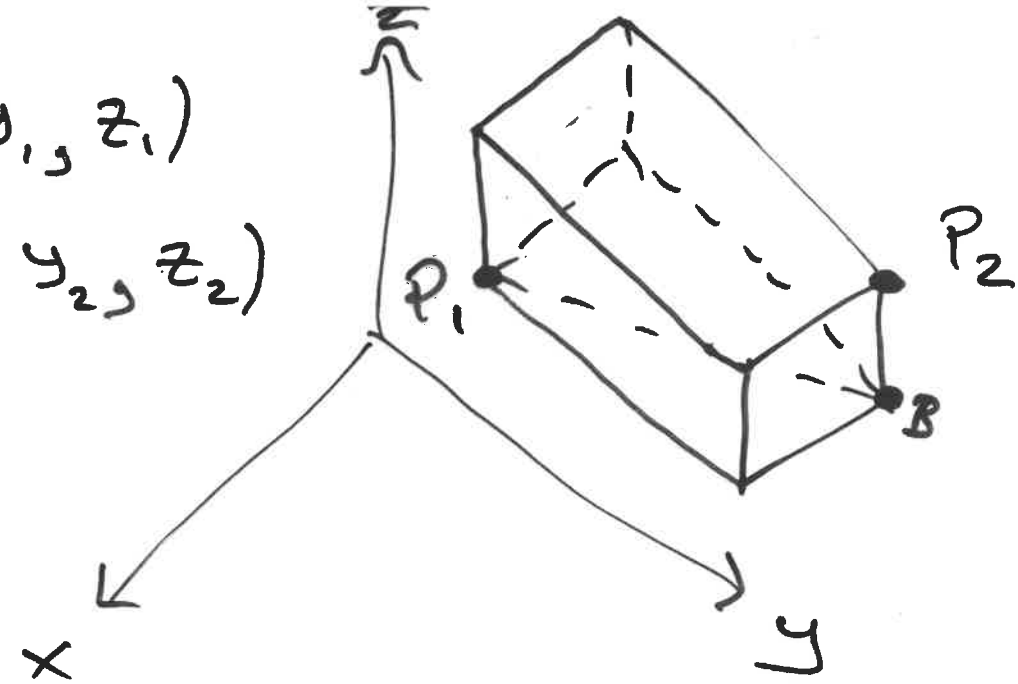
Distance formula in three dimensions

The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$P_1(x_1, y_1, z_1)$

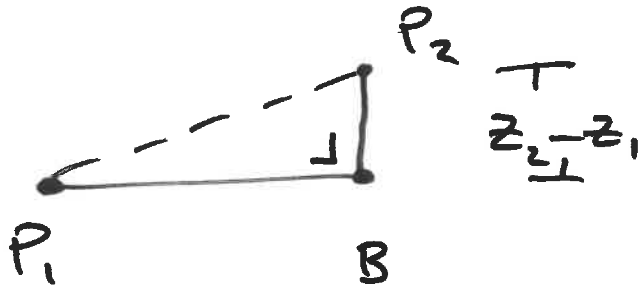
$P_2(x_2, y_2, z_2)$



P_1B is parallel to the xy plane

$P_1(x_1, y_1, z_1) \quad B(x_2, y_2, z_1)$

$$|P_1B| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$



So $|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$

or $|P_1P_2|^2 = \left[\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}\right]^2 + (z_2-z_1$

or $|P_1P_2|^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2$

or $|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

Example:

What is the distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$?

$$|PQ| = \sqrt{(1-2)^2 + (-3-(-1))^2 + (5-7)^2}$$
$$= \sqrt{(-1)^2 + (-3+1)^2 + (-2)^2}$$
$$= \sqrt{1 + (-2)^2 + 4}$$
$$= \sqrt{1 + 4 + 4}$$
$$= \sqrt{9} = 3.$$

Exercise:

Find the lengths of the sides of the triangle PQR with $P(3, -2, -3)$, $Q(7, 0, 1)$ and $R(1, 2, 1)$.

Is it a right-angle triangle?

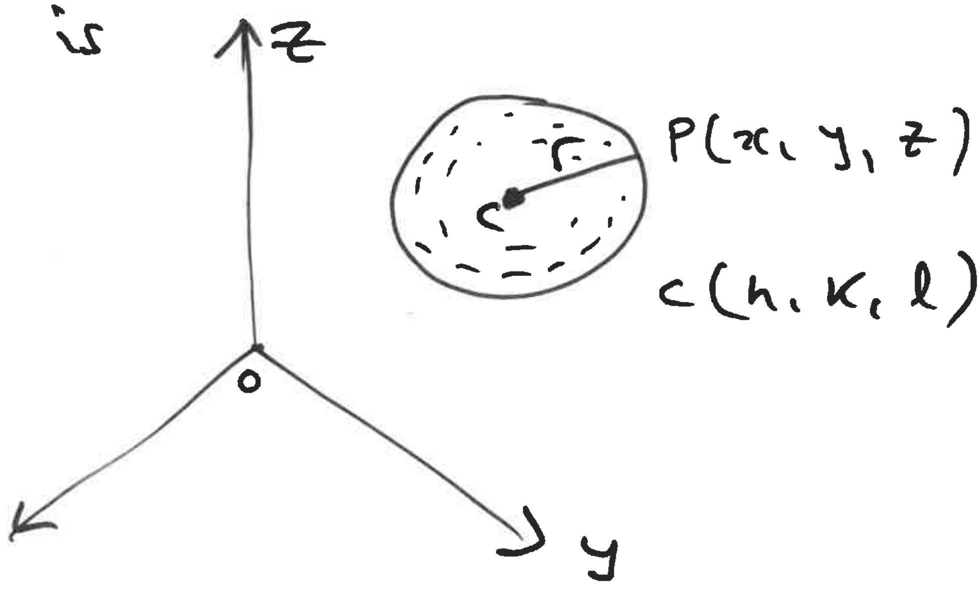
Is it an isosceles triangle?

Equation of a Sphere

An equation of a sphere with centre $C(h, k, l)$ and radius r is:

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

Its surface is



Completing the square

e.g. $x^2 + 6x$

can be written as

$$x^2 + 6x + \left(\frac{1}{2}\text{ coefficient of } x\right)^2 - \left(\frac{1}{2}\text{ coefficient of } x\right)^2$$

adding + subtracting the same

$$= x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2$$

$$= x^2 + 6x + 3^2 - 3^2$$
$$= (x^2 + 6x + 9) - 9$$
$$= (x+3)^2 - 9$$

completed the square

Example:

Show that

$$x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$$

is the equation of a sphere, and find its centre and radius.

Solution:

we write the equation as:

$$(x^2 + 4x) + (y^2 - 6y) + (z^2 + 2z) = -6$$

or $\left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right) - \left(\frac{4}{2}\right)^2$
$+ \left(y^2 - 6y + \left(\frac{-6}{2}\right)^2\right) - \left(\frac{-6}{2}\right)^2$
$+ \left(z^2 + 2z + \left(\frac{2}{2}\right)^2\right) - \left(\frac{2}{2}\right)^2 = -6$

OR $(x^2 + 4x + 4) - 4$
$+ (y^2 - 6y + 9) - 9$
$+ (z^2 + 2z + 1) - 1 = -6$

OR $(x+2)^2 - 4$

$+ (y-3)^2 - 9$

$+ (z+1)^2 - 1 = -6$

OR $(x+2)^2 + (y-3)^2 + (z+1)^2 = 4+9+1-6$

or $(x+2)^2 + (y-3)^2 + (z+1)^2 = 8$

Compare this with the equation of a sphere:

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

$C(h, k, l)$ is $\underline{(-2, 3, -1)}$

$r^2 = 8$ so $r = \sqrt{8}$ $\left(= \sqrt{4.2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}\right)$

Example:

Find an equation of the sphere that passes through the point $(4, 3, -1)$ and has centre $(3, 8, 1)$.

Solution.

To get the equation, we need (a) centre $\leftarrow$ given

(b) radius

As the radius is the distance between the centre and any point on the sphere

$\rightarrow$ the radius is the distance between $(3,8,1)$ and $(4,3,-1)$.

$$= \sqrt{(4-3)^2 + (3-8)^2 + (-1-1)^2}$$
$$= \sqrt{1^2 + (-5)^2 + (-2)^2}$$
$$= \sqrt{1+25+4} = \sqrt{30}.$$

The equation is:

$$(x-3)^2 + (y-8)^2 + (z-1)^2 = 30.$$

Example:

What region in $\mathbb{R}^3$ is represented by the following inequalities?

$$\begin{cases} 1 \leq x^2+y^2+z^2 \leq 4 \\ z \leq 0 \end{cases}$$

Solution

$$x^2+y^2+z^2=1$$

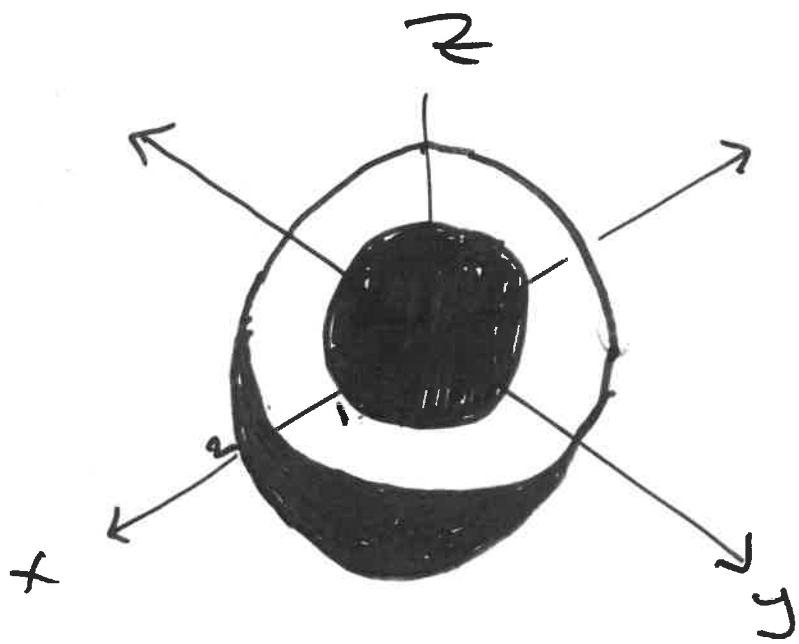
is the sphere with centre the origin and radius 1

$$x^2+y^2+z^2=4$$

is the sphere with centre the origin and radius 2

→ Region is the space between the two spheres

→ As $z \leq 0$ → its the part of this space below the xy plane i.e.



Exercise:

Write inequalities to describe the region consisting of all points between (but not on) the spheres of radius 2 and 7 centered at the origin.

Exercise

Sketch: $x^2+y^2=16$ in $\mathbb{R}^3$.

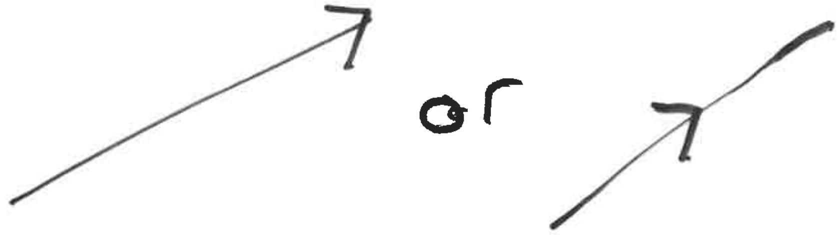
VECTORS

The term vector is used by scientists to indicate a quantity (such as velocity or force) that has both

1. Magnitude

and 2. direction.

+ A vector is often represented by an arror or a directed line segment.

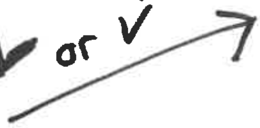


→ The length of the arrow represents the <u>magnitude</u> of the vector and the arrow points in the <u>direction</u> of the vector.

→ We denote a vector by:

- printing a letter in boldface **v**
- or by putting an arrow above the letter $\vec{v}$

v or $\vec{v}$



NOTE: A vector can be moved around space

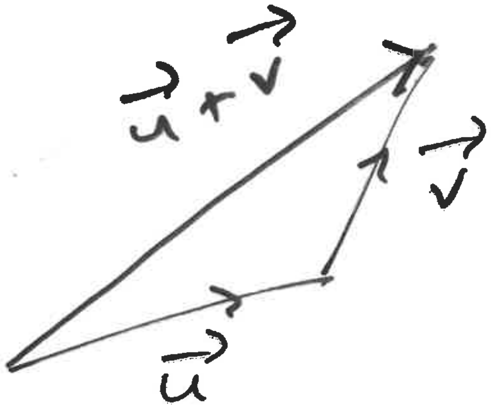
If we start with vectors $\vec{u}$ and $\vec{v}$,
we first move $\vec{v}$ so that its tail coincides with the tip of $\vec{u}$ and define the sum of $\vec{u}$ and $\vec{v}$ as follows:

<u>Definition</u> (Vector addition)

If $\vec{u}$ and $\vec{v}$ are vectors positioned so that the initial point of $\vec{v}$ is at the terminal point of $\vec{u}$, then the sum of $\vec{u}+\vec{v}$ is the vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$.

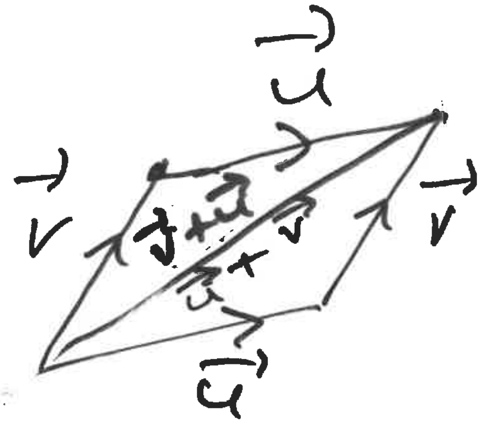
This is known as the <u>Triangle law</u>

i.e.



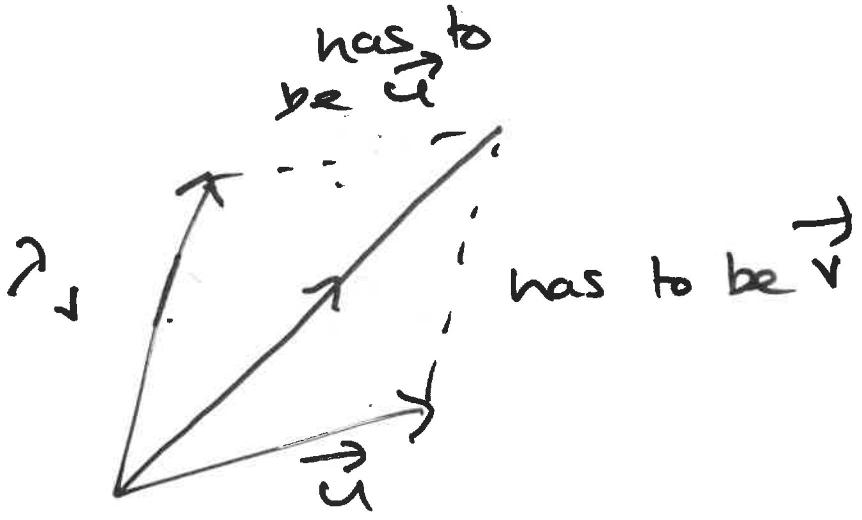
Start with the same vectors $\vec{u}$ and $\vec{v}$ as before and draw another copy of $\vec{v}$ whose tip is at the initial point of $\vec{u}$.

Completing the parallelogram, we see $\vec{u}+\vec{v} = \vec{v}+\vec{u}$



This gives another way to construct the sum:

→ if we place $\vec{u}$ and $\vec{v}$ so they start at the same point, then $\vec{u}+\vec{v}$ lies along the diagonal of the parallelogram with $\vec{u}$ and $\vec{v}$ as sides.



This is known as the <u>parallelogram</u> <u>law</u>.

We can multiply a vector by a real number c.

This real number is called a <u>scalar</u>.

<u>Definition</u> (Scalar multiplication)

If c is a scalar and $\vec{v}$ is a vector, then the scalar multiple $c\vec{v}$ is the vector

- whose length is $|c|$ times the length of $\vec{v}$
- and whose direction is the same as $\vec{v}$ if $c>0$ and opposite if $c<0$.

If $c=0$ or $\vec{v}=\vec{0}$, then $c\vec{v}=\vec{0}$.

Note:

→ Two nonzero vectors are <u>parallel</u> if they are scalar multiples of one another.

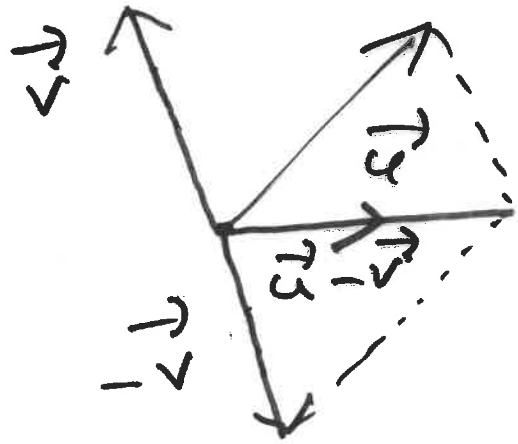
→ The vector $-\vec{v} = (-1)\vec{v}$ has the same length as $\vec{v}$ but points in the opposite direction. We call it the <u>negative</u> of $\vec{v}$.

→ By the <u>difference</u> of two vectors, we mean

$$\vec{u} - \vec{v} = u + (-\vec{v}),$$

Hence we can construct $\vec{u} - \vec{v}$ by first drawing the negative of $\vec{v}$, $-\vec{v}$

and then adding it to $\vec{u}$ by the parallelogram law

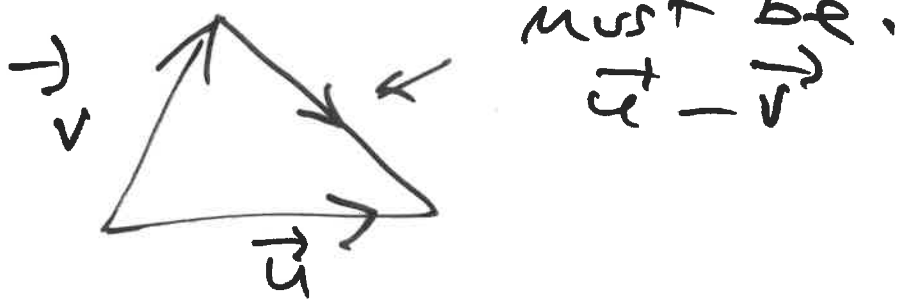


we can also think of the difference as follows:

As $\vec{v}+(\vec{u}-\vec{v}) = \vec{u}$

i.e. adding $\vec{v}$ and $\vec{u}-\vec{v}$ we get $\vec{u}$

geometrically, using the triangle law



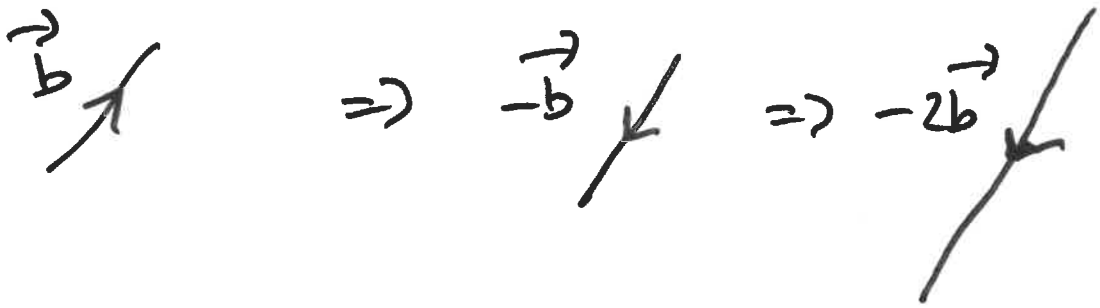
Example:

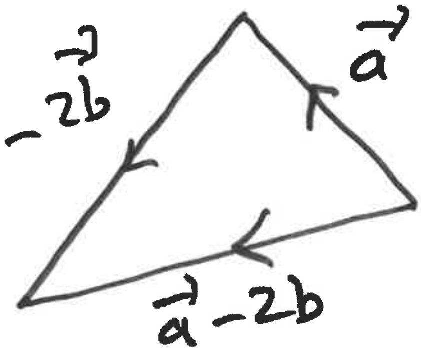
If $\vec{a}$ and $\vec{b}$ are the two vectors below, draw $\vec{a} - 2\vec{b}$.

<u>Solution</u>

we have to add $\vec{a}$ and $-2\vec{b}$

→ So place $-2\vec{b}$ tail at the end of $\vec{a}$



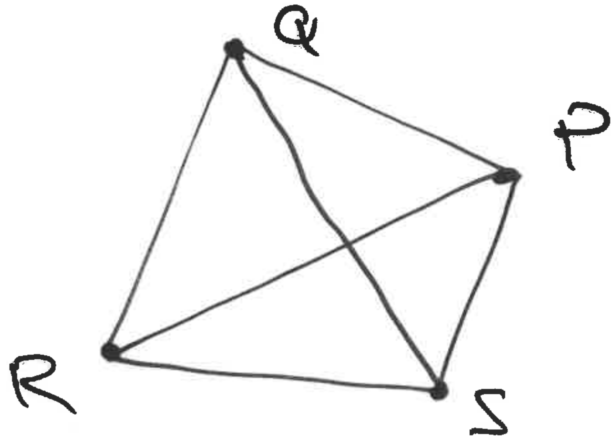


Exercise

- Write each combination of vectors as a single vector

① $\overrightarrow{QS} - \overrightarrow{PS}$

② $\overrightarrow{QR} - \overrightarrow{SR}$



- Use the vectors below, draw

① $\vec{a} + \vec{b}$

② $\vec{a} - \vec{b}$

③ $2\vec{a}$

④ $-\frac{1}{2}\vec{b}$

⑤ $2\vec{a} + \vec{b}$

⑥ $\vec{b} - 3\vec{a}$

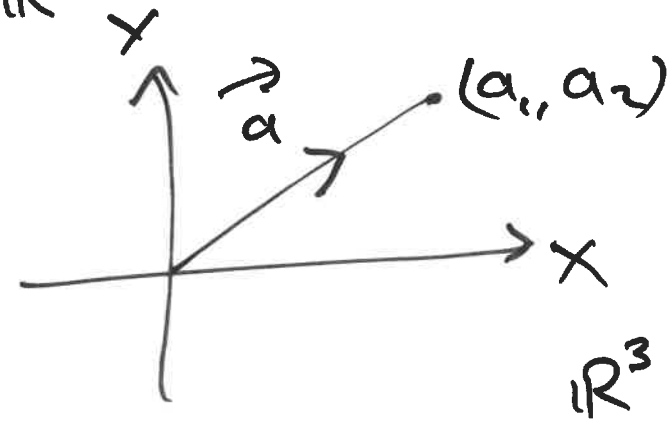
Components

If the initial point of vector $\vec{a}$ is placed at the origin of a rectangular coordinate system, then the terminal point of $\vec{a}$ has coordinates:

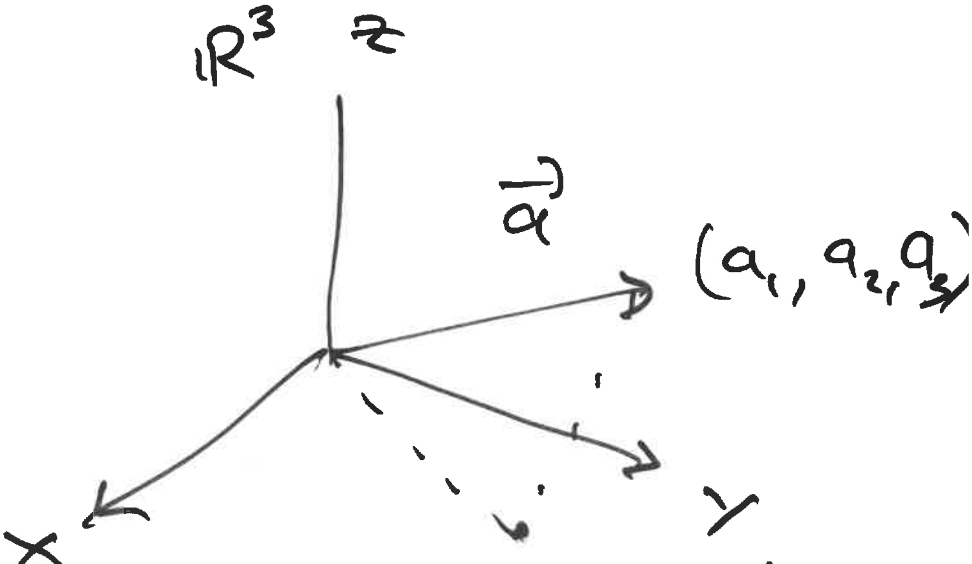
(a_1, a_2) in $\mathbb{R}^2$

(a_1, a_2, a_3) in $\mathbb{R}^3$.

$\mathbb{R}^2$



$\mathbb{R}^3$



These coordinates are called

The components are denoted by

$$\vec{a} = \langle a_1, a_2 \rangle \text{ in } \mathbb{R}^2$$

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \text{ in } \mathbb{R}^3.$$

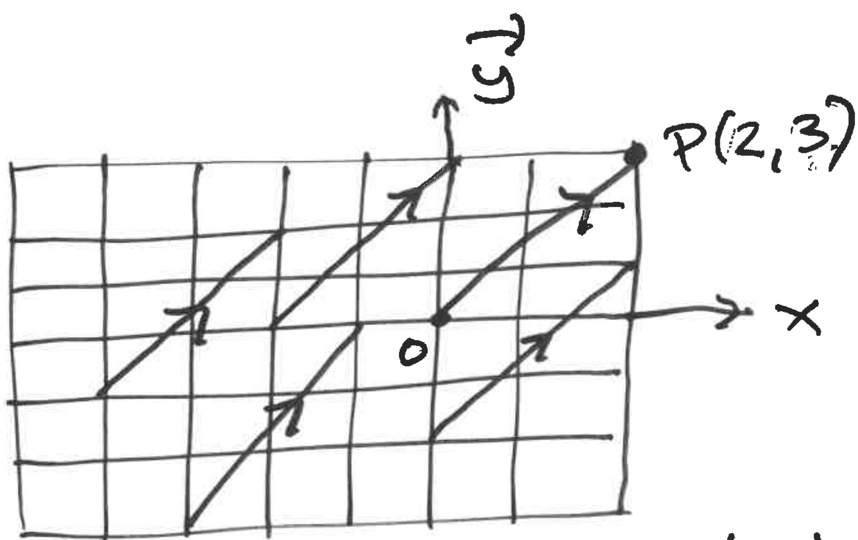
So (a_1, a_2) denotes a point in a plane

$\langle a_1, a_2 \rangle$ denotes a vector

and (a_1, a_2, a_3) denotes a point in space

$\langle a_1, a_2, a_3 \rangle$ denotes a vector

Example



All these vectors are equivalent to $\vec{OP} = \langle 2, 3 \rangle$ but there is one $(2, 3)$

Common feature:

the terminal point is reached from the initial point by a displacement of two units to the right and three upwards

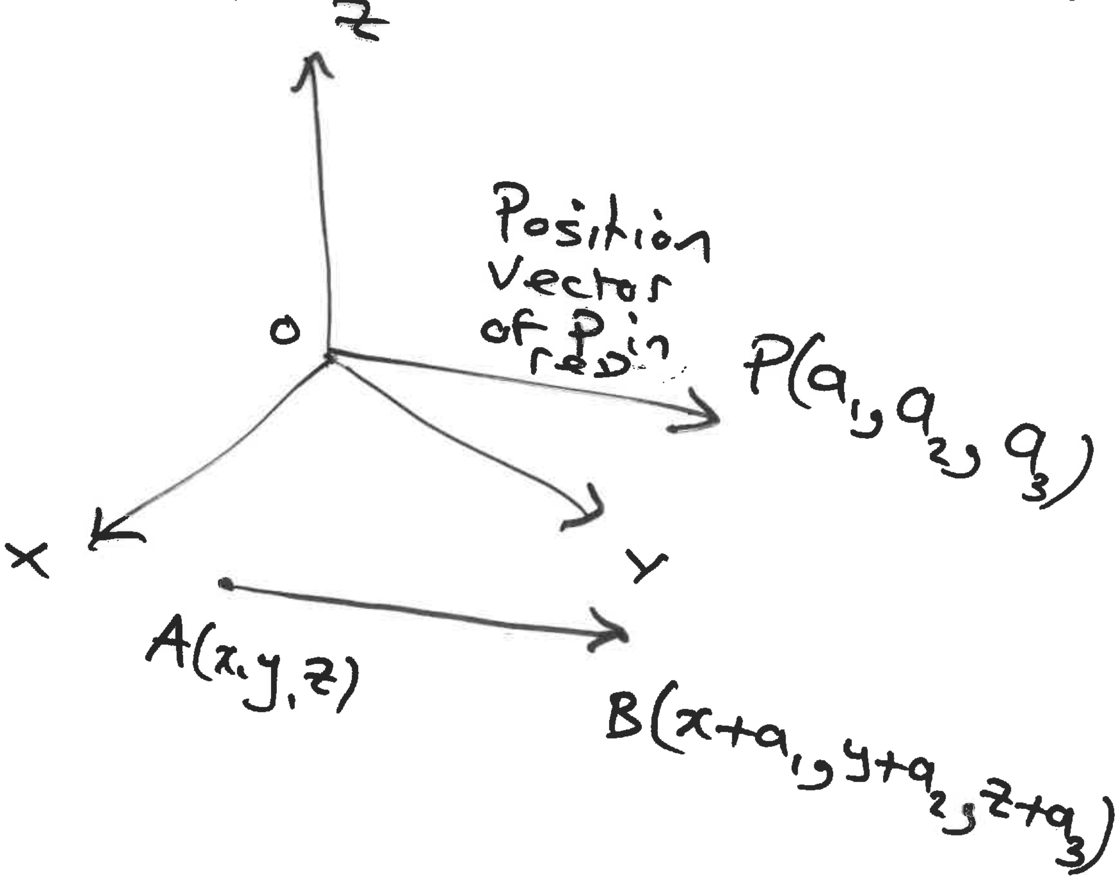
→ These vectors (in red) are representations of the algebraic vector $\vec{a} = \langle 2, 3 \rangle$

→ The particular representation $\overrightarrow{OP}$ from the origin to point $P(3,2)$ (in blue) is called the position vector of the point P.

The vector $\vec{a} = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$

is the <u>position vector</u>

of the point $P(a_1, a_2, a_3)$



For any other representation

$\overrightarrow{AB}$ of $\vec{a}$

when the initial point is:

$A(x_1, y_1, z_1)$

and the terminal point is:

$B(x_2, y_2, z_2)$

we must have

$x_1 + a_1 = x_2$, $y_1 + a_2 = y_2$, $z_1 + a_3 = z_2$.

or $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, $a_3 = z_2 - z_1$

Given the points

$A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

The vector $\vec{a}$ with representation $\overrightarrow{AB}$ is:

$$\vec{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Example,

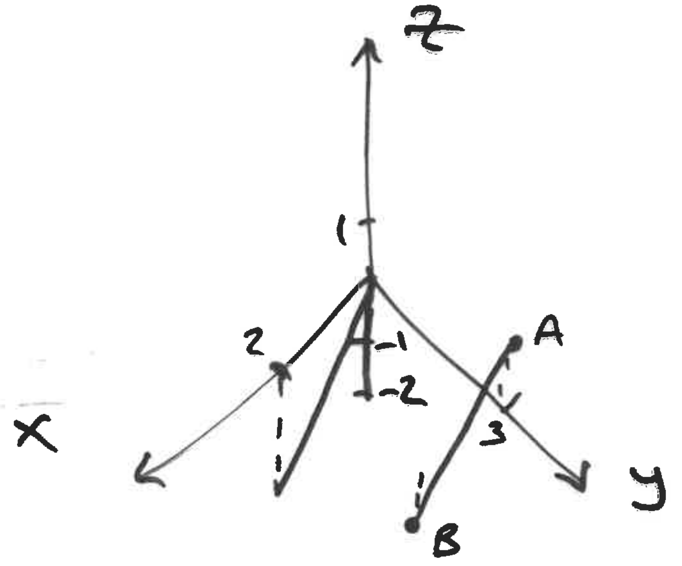
Let $A(0, 3, 1)$ and $B(2, 3, -1)$.

Find a vector $\vec{a}$ with representation given by the directed line segment $\overrightarrow{AB}$.

Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.

Solution

$$\vec{a} = \langle 2-0, 3-3, -1-1 \rangle$$
$$= \langle 2, 0, -2 \rangle$$



Note in both cases:

x goes up 2
y stays the same
z goes down 2

Magnitude of a vector

The magnitude or length of the vector is the length of any of its representations and is denoted by the symbol

$$|\vec{v}| \quad \text{or} \quad \|\vec{v}\|$$

Using the distance formula to compute the length of a line segment OP, we obtain:

- the length of a two-dimensional vector $\vec{a} = \langle a_1, a_2 \rangle$ is

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2}$$

- the length of a three-dimensional vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is

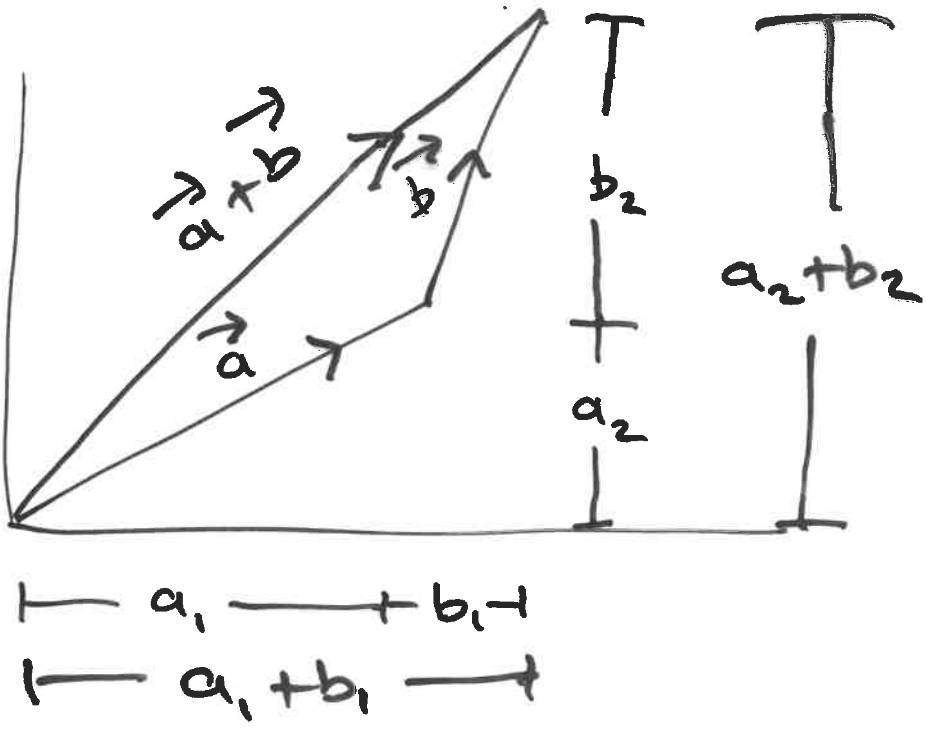
$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Adding vectors algebraically

if $\vec{a} = \langle a_1, a_2 \rangle$

and $\vec{b} = \langle b_1, b_2 \rangle$, then

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$



If $\vec{a} = \langle a_1, a_2 \rangle$ $\vec{b} = \langle b_1, b_2 \rangle$

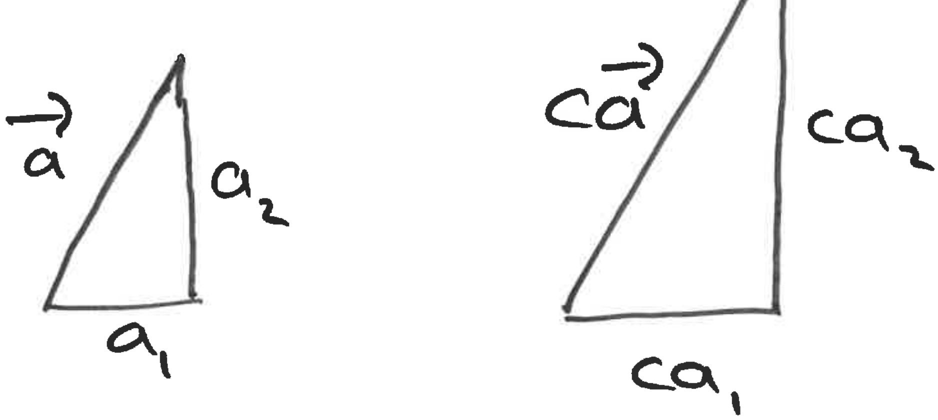
then

$$\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

If $\vec{a} = \langle a_1, a_2 \rangle$

and c is a scalar, then

$$c\vec{a} = \langle ca_1, ca_2 \rangle$$



If $\vec{a} = \langle a_1, a_2, a_3 \rangle$,
$\vec{b} = \langle b_1, b_2, b_3 \rangle$
and c is a scalar, then

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$$

Exercise

if $\vec{a} = \langle 4, 0, 3 \rangle$ and
$\vec{b} = \langle -2, 1, 5 \rangle$

1. $|\vec{a}|$
2. $\vec{a} + \vec{b}$
3. $\vec{a} - \vec{b}$
4. $3\vec{b}$
5. $2\vec{a} + 5\vec{b}$

Properties of Vectors

1. $\vec{a}+\vec{b} = \vec{b}+\vec{a}$
2. $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$
3. $\vec{a}+\vec{0} = \vec{a}$
4. $\vec{a}+(-\vec{a}) = \vec{0}$
5. $c(\vec{a}+\vec{b}) = c\vec{a}+c\vec{b}$
6. $(c+d)\vec{a} = c\vec{a}+d\vec{a}$
7. $(cd)\vec{a} = c(d\vec{a})$
8. $1\vec{a} = \vec{a}$